Useful Inequalities $\{x^2 \geqslant 0\}$

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Cauchy-Schwarz

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Minkowski

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}} \quad \text{for } p \ge 1.$$

Hölder

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q} \quad \text{for } p,q > 1,\ \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Bernoulli

$$(1+x)^r \ge 1 + rx \quad \text{for } x \ge -1,\ r \in \mathbb{R} \setminus (0,1).\ \text{Reverse for } r \in [0,1].$$

$$(1+x)^r \le \tfrac{1}{1-rx} \quad \text{for } x \in [-1, \tfrac{1}{r}),\quad r \ge 0.$$

$$(1+x)^r \le 1 + \tfrac{x}{x+1} r \quad \text{for } x \ge 0,\quad r \in [-1, 0].$$

$$(1+x)^r \le 1 + (2^r - 1)x \quad \text{for } x \in [0,1],\quad r \in \mathbb{R} \setminus (0,1).$$

$$(1+nx)^{n+1} \ge (1+(n+1)x)^n \quad \text{for } x \ge 0,\ n \in \mathbb{N}.$$

$$(a+b)^n \le a^n + nb(a+b)^{n-1} \quad \text{for } a,b \ge 0,\ n \in \mathbb{N}.$$

exponential

$$e^x \ge \left(1 + \tfrac{x}{n}\right)^n \ge 1 + x; \quad \left(1 + \tfrac{x}{n}\right)^n \ge e^x\left(1 - \tfrac{x^2}{n}\right) \quad \text{for } n \ge 1,\ |x| \le n.$$

$$\tfrac{x^n}{n!} + 1 \le e^x \le \left(1 + \tfrac{x}{n}\right)^{n+x/2}; \quad e^x \ge \left(\tfrac{ex}{n}\right)^n \quad \text{for } x, n > 0.$$

$$x^y + y^x > 1;\ x^y > \tfrac{x}{x+y};\ e^x > \left(1 + \tfrac{x}{y}\right)^y > e^{\frac{xy}{x+y}};\ \tfrac{x}{y} \ge e^{\frac{x-y}{x}} \quad \text{for } x,y > 0.$$

$$\tfrac{1}{2-x} < x^x < x^2 - x + 1; \quad e^{2x} \le \tfrac{1+x}{1-x} \quad \text{for } x \in (0,1).$$

$$x^{1/r}(x-1) \le rx(x^{1/r} - 1) \quad \text{for } x, r \ge 1; \quad 2^{-x} \le 1 - \tfrac{x}{2} \quad \text{for } x \in [0,1].$$

$$xe^x \ge x + x^2 + \tfrac{x^3}{2}; \quad e^x \le x + e^{x^2}; \quad e^x + e^{-x} \le 2e^{x^2/2} \quad \text{for } x \in \mathbb{R}.$$

$$e^{-x} \le 1 - \tfrac{x}{2} \quad \text{for } x \in [0, 1.59]; \quad e^x \le 1 + x + x^2 \quad \text{for } x < 1.79.$$

$$\left(1 + \tfrac{x}{p}\right)^p \ge \left(1 + \tfrac{x}{q}\right)^q \quad \text{for } (i)\ x > 0,\ p > q > 0,$$

$(ii)\ -p < -q < x < 0,\ (iii)\ -q > -p > x > 0.$ Reverse for:

$(iv)\ q < 0 < p\,,\ -q > x > 0,\ (v)\ q < 0 < p\,,\ -p < x < 0.$

logarithm

$$\tfrac{x}{1+x} \le \ln(1+x) \le \tfrac{x(6+x)}{6+4x} \le x \quad \text{for } x > -1.$$

$$\tfrac{2}{2+x} \le \tfrac{1}{\sqrt{1+x+x^2/12}} \le \tfrac{\ln(1+x)}{x} \le \tfrac{1}{\sqrt{x+1}} \le \tfrac{2+x}{2+2x} \quad \text{for } x > -1.$$

$$\ln(n) + \tfrac{1}{n+1} < \ln(n+1) < \ln(n) + \tfrac{1}{n} \le \textstyle\sum_{i=1}^{n} \tfrac{1}{i} \le \ln(n) + 1 \quad \text{for } n \ge 1.$$

$$|\ln x| \le \tfrac{1}{2}\left|x - \tfrac{1}{x}\right|;\ \ln(x+y) \le \ln(x) + \tfrac{y}{x};\ \ln x \le y(x^{\frac{1}{y}} - 1);\ x, y > 0.$$

$$\ln(1+x) \ge x - \tfrac{x^2}{2} \quad \text{for } x \ge 0; \quad \ln(1+x) \ge x - x^2 \quad \text{for } x \ge -0.68.$$

trigonometric

$$x - \tfrac{x^3}{2} \le x\cos x \le \tfrac{x\cos x}{1-x^2/3} \le x\sqrt[3]{\cos x} \le x - x^3/6 \le x\cos\tfrac{x}{\sqrt{3}} \le \sin x,$$

hyperbolic

$$x\cos x \le \tfrac{x^3}{\sinh^2 x} \le x\cos^2(x/2) \le \sin x \le (x\cos x + 2x)/3 \le \tfrac{x^2}{\sinh x},$$

$$\max\left\{\tfrac{2}{\pi}, \tfrac{\pi^2 - x^2}{\pi^2 + x^2}\right\} \le \tfrac{\sin x}{x} \le \cos\tfrac{x}{2} \le 1 \le 1 + \tfrac{x^2}{3} \le \tfrac{\tan x}{x} \quad \text{for } x \in \left[0, \tfrac{\pi}{2}\right].$$

square root

$$2\sqrt{x+1} - 2\sqrt{x} < \tfrac{1}{\sqrt{x}} < \sqrt{x+1} - \sqrt{x-1} < 2\sqrt{x} - 2\sqrt{x-1} \quad \text{for } x \ge 1.$$

$$x \le \tfrac{x+1}{2} - \tfrac{(x-1)^2}{2} \le \sqrt{x} \le \tfrac{x+1}{2} - \tfrac{(x-1)^2}{8} \quad \text{for } x \in [0,1].$$

binomial

$$\max\left\{\tfrac{n^k}{k^k}, \tfrac{(n-k+1)^k}{k!}\right\} \le \tbinom{n}{k} \le \tfrac{n^k}{k!} \le \left(\tfrac{en}{k}\right)^k; \quad \tbinom{n}{k} \le \tfrac{n^n}{k^k(n-k)^{n-k}}.$$

$$\tfrac{n^k}{4k!} \le \tbinom{n}{k} \quad \text{for } \sqrt{n} \ge k \ge 0; \quad \tfrac{4^n}{\sqrt{\pi n}}\left(1 - \tfrac{1}{8n}\right) \le \tbinom{2n}{n} \le \tfrac{4^n}{\sqrt{\pi n}}\left(1 - \tfrac{1}{9n}\right).$$

$$\tbinom{n_1}{k_1}\tbinom{n_2}{k_2} \le \tbinom{n_1+n_2}{k_1+k_2}; \quad \tbinom{tn}{k} \ge t^k \tbinom{n}{k} \quad \text{for } t \ge 1.$$

$$\tfrac{\sqrt{\pi}}{2} G \le \tbinom{n}{\alpha n} \le G \quad \text{for } G = \tfrac{2^{nH(\alpha)}}{\sqrt{2\pi n\alpha(1-\alpha)}},\ H(x) = -\log_2(x^x(1-x)^{1-x}).$$

$$\textstyle\sum_{i=0}^{d} \tbinom{n}{i} \le \min\left\{n^d + 1,\ \left(\tfrac{en}{d}\right)^d,\ 2^n\right\} \quad \text{for } n \ge d \ge 1.$$

$$\textstyle\sum_{i=0}^{\alpha n} \tbinom{n}{i} \le \min\left\{\tfrac{1-\alpha}{1-2\alpha}\tbinom{n}{\alpha n},\ 2^{nH(\alpha)},\ 2^n e^{-2n\left(\frac{1}{2} - \alpha\right)^2}\right\} \quad \text{for } \alpha \in (0, \tfrac{1}{2}).$$

binary entropy

$$4x(1-x) \le H(x) \le (4x(1-x))^{1/\ln 4}; \quad \tfrac{H(x^2)}{H(x)} \ge 1.618x \text{ for } x \in (0,1).$$

$$1 - 5x^2 \le H(1/2 - x) \le 1 - x^2, \text{ for } 0 < x \le 1/4.$$

Stirling

$$e\left(\tfrac{n}{e}\right)^n \le \sqrt{2\pi n}\left(\tfrac{n}{e}\right)^n e^{1/(12n+1)} \le n! \le \sqrt{2\pi n}\left(\tfrac{n}{e}\right)^n e^{1/12n} \le en\left(\tfrac{n}{e}\right)^n$$

means

$$\min x_i \le \tfrac{n}{\sum x_i^{-1}} \le \left(\textstyle\prod x_i\right)^{1/n} \le \tfrac{1}{n}\textstyle\sum x_i \le \sqrt{\tfrac{1}{n}\textstyle\sum {x_i}^2} \le \tfrac{\sum x_i^2}{\sum x_i} \le \max x_i$$

$$\min_i \tfrac{|x_i|}{|y_i|} \le \left(\textstyle\sum_i |x_i|\right) / \left(\textstyle\sum_i |y_i|\right) \le \max_i \tfrac{|x_i|}{|y_i|}$$

power means

$M_p \le M_q$ for $p \le q$, where $M_p = \left(\sum_i w_i |x_i|^p\right)^{1/p}$, $w_i \ge 0$, $\sum_i w_i = 1$.
In the limit $M_0 = \prod_i |x_i|^{w_i}$, $M_{-\infty} = \min_i\{x_i\}$, $M_\infty = \max_i\{x_i\}$.

Lehmer

$$\frac{\sum_i w_i |x_i|^p}{\sum_i w_i |x_i|^{p-1}} \le \frac{\sum_i w_i |x_i|^q}{\sum_i w_i |x_i|^{q-1}} \quad \text{for } p \le q,\quad w_i \ge 0.$$

log mean

$$\sqrt{xy} \le \left(\tfrac{\sqrt{x}+\sqrt{y}}{2}\right)(xy)^{\frac{1}{4}} \le \tfrac{x-y}{\ln(x)-\ln(y)} \le \left(\tfrac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \le \tfrac{x+y}{2} \quad \text{for } x, y > 0.$$

Heinz

$$\sqrt{xy} \le \tfrac{x^{1-\alpha}y^\alpha + x^\alpha y^{1-\alpha}}{2} \le \tfrac{x+y}{2} \quad \text{for } x, y > 0,\quad \alpha \in [0,1].$$

Maclaurin-Newton

$${S_k}^2 \ge S_{k-1}S_{k+1} \quad \text{and} \quad (S_k)^{1/k} \ge (S_{k+1})^{1/(k+1)} \quad \text{for } 1 \le k < n,$$

$$S_k = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1} a_{i_2} \cdots a_{i_k}, \quad \text{and} \quad a_i \ge 0.$$

Jensen

$$\varphi\left(\textstyle\sum_i p_i x_i\right) \le \textstyle\sum_i p_i \varphi(x_i) \quad \text{where } p_i \ge 0,\ \textstyle\sum p_i = 1,\ \text{and } \varphi \text{ convex.}$$

Alternatively: $\varphi(\mathrm{E}[X]) \le \mathrm{E}[\varphi(X)]$. For concave φ the reverse holds.

Chebyshev

$$\sum_{i=1}^{n} f(x_i) g(x_i) p_i \ge \left(\sum_{i=1}^{n} f(x_i) p_i\right)\left(\sum_{i=1}^{n} g(x_i) p_i\right)$$

for $x_1 \le \cdots \le x_n$ and f, g nondecreasing, $p_i \ge 0$, $\sum p_i = 1$.
Alternatively: $\mathrm{E}\big[f(X)g(X)\big] \ge \mathrm{E}\big[f(X)\big]\mathrm{E}\big[g(X)\big]$.

rearrangement

$$\sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} a_i b_{\pi(i)} \ge \sum_{i=1}^{n} a_i b_{n-i+1} \quad \text{for } a_1 \le \cdots \le a_n,$$

$b_1 \le \cdots \le b_n$ and π a permutation of $[n]$. More generally:

$$\sum_{i=1}^{n} f_i(b_i) \ge \sum_{i=1}^{n} f_i(b_{\pi(i)}) \ge \sum_{i=1}^{n} f_i(b_{n-i+1})$$

with $\big(f_{i+1}(x) - f_i(x)\big)$ nondecreasing for all $1 \le i < n$.

Dually: $\displaystyle\prod_{i=1}^{n}(a_i + b_i) \le \prod_{i=1}^{n}(a_i + b_{\pi(i)}) \le \prod_{i=1}^{n}(a_i + b_{n-i+1}) \quad$ for $a_i, b_i \ge 0$.

Weierstrass $1 - \sum_i w_i x_i \le \prod_i \big(1 - x_i\big)^{w_i}$, and

$1 + \sum_i w_i x_i \le \prod_i \big(1 + x_i\big)^{w_i} \le \prod_i \big(1 - x_i\big)^{-w_i}$ for $x_i \in [0,1]$, $w_i \ge 1$.

$\prod_i \big(1 \pm x_i\big)^{w_i} \le \big(1 \mp \sum_i w_i x_i\big)^{-1}$ if additionally $\sum_i w_i x_i \le 1$.

Kantorovich $\left(\sum_i {x_i}^2\right)\left(\sum_i {y_i}^2\right) \le \left(\frac{A}{G}\right)^2 \left(\sum_i x_i y_i\right)^2$ for $x_i, y_i > 0$,

$0 < m \le \frac{x_i}{y_i} \le M < \infty$, $A = (m+M)/2$, $G = \sqrt{mM}$.

Nesbitt $\sum_{i=1}^n \frac{a_i}{S - a_i} \ge \frac{n}{n-1}$ and $\sum_{i=1}^n \frac{S}{S - a_i} \ge \frac{n^2}{n-1}$ for $a_i \ge 0$, $S = \sum a_i$.

sum & integral $\int_{L-1}^{U} f(x)\,dx \le \sum_{i=L}^{U} f(i) \le \int_{L}^{U+1} f(x)\,dx$ for f nondecreasing.

Cauchy $f'(a) \le \frac{f(b)-f(a)}{b-a} \le f'(b)$ where $a < b$, and f convex.

Hermite $\varphi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b \varphi(x)\,dx \le \frac{\varphi(a)+\varphi(b)}{2}$ for φ convex.

Gibbs $\sum_i a_i \log \frac{a_i}{b_i} \ge a \log \frac{a}{b}$ for $a_i, b_i \ge 0$, or more generally:

$\sum_i a_i \varphi\big(\frac{b_i}{a_i}\big) \le a\, \varphi\big(\frac{b}{a}\big)$ for φ concave, and $a = \sum a_i$, $b = \sum b_i$.

Chong $\sum\limits_{i=1}^{n} \frac{a_i}{a_{\pi(i)}} \ge n$ and $\prod\limits_{i=1}^{n} {a_i}^{a_i} \ge \prod\limits_{i=1}^{n} {a_i}^{a_{\pi(i)}}$ for $a_i > 0$.

Schur-Vornicu $f(x)(x-y)^k(x-z)^k + f(y)(y-z)^k(y-x)^k + f(z)(z-x)^k(z-y)^k \ge 0$

where $x, y, z \ge 0$, $k \ge 1$ integer, f convex or monotonic, $f \ge 0$.

Young $\left(\frac{1}{px^p} + \frac{1}{qy^q}\right)^{-1} \le xy \le \frac{x^p}{p} + \frac{y^q}{q}$ for $x, y, p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$.

$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \ge ab$, for f cont., strictly increasing.

Shapiro $\sum\limits_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} \ge \frac{n}{2}$ where $x_i > 0$, $(x_{n+1}, x_{n+2}) := (x_1, x_2)$,

and $n \le 12$ if even, $n \le 23$ if odd.

Hadamard $(\det A)^2 \le \prod\limits_{i=1}^{n} \sum\limits_{j=1}^{n} A_{ij}^2$ where A is an $n \times n$ matrix.

Schur $\sum_{i=1}^n \lambda_i^2 \le \sum_{i=1}^n \sum_{j=1}^n A_{ij}^2$ and $\sum_{i=1}^k d_i \le \sum_{i=1}^k \lambda_i$ for $1 \le k \le n$.

A is an $n \times n$ matrix. For the second inequality A is symmetric.

$\lambda_1 \ge \cdots \ge \lambda_n$ the eigenvalues, $d_1 \ge \cdots \ge d_n$ the diagonal elements.

Ky Fan $\dfrac{\prod_{i=1}^n x_i^{a_i}}{\prod_{i=1}^n (1-x_i)^{a_i}} \le \dfrac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i (1-x_i)}$ for $x_i \in [0, \frac{1}{2}]$, $a_i \in [0,1]$, $\sum a_i = 1$.

Aczél $\big(a_1 b_1 - \sum_{i=2}^n a_i b_i\big)^2 \ge \big(a_1^2 - \sum_{i=2}^n a_i^2\big)\big(b_1^2 - \sum_{i=2}^n b_i^2\big)$

given that $a_1^2 > \sum_{i=2}^n a_i^2$ or $b_1^2 > \sum_{i=2}^n b_i^2$.

Mahler $\prod\limits_{i=1}^{n} \big(x_i + y_i\big)^{1/n} \ge \prod\limits_{i=1}^{n} x_i^{1/n} + \prod\limits_{i=1}^{n} y_i^{1/n}$ where $x_i, y_i > 0$.

Abel $b_1 \cdot \min\limits_k \sum\limits_{i=1}^{k} a_i \le \sum\limits_{i=1}^{n} a_i b_i \le b_1 \cdot \max\limits_k \sum\limits_{i=1}^{k} a_i$ for $b_1 \ge \cdots \ge b_n \ge 0$.

Milne $\left(\sum_{i=1}^n (a_i + b_i)\right)\left(\sum_{i=1}^n \frac{a_i b_i}{a_i + b_i}\right) \le \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right)$ for $a_i, b_i \ge 0$.

Carleman $\sum_{k=1}^n \left(\prod_{i=1}^k |a_i|\right)^{1/k} \le e \sum_{k=1}^n |a_k|$

sum & product $\left|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i\right| \le \sum_{i=1}^n |a_i - b_i|$ for $|a_i|, |b_i| \le 1$.

$\prod_{i=1}^n (t + a_i) \ge (t+1)^n$ where $\prod_{i=1}^n a_i \ge 1$, $a_i > 0$, $t > 0$.

Radon $\sum_i \frac{x_i^p}{a_i^{p-1}} \ge \frac{(\sum_i x_i)^p}{(\sum_i a_i)^{p-1}}$ for $x_i, a_i \ge 0$, $p \ge 1$ (rev. if $p \in [0,1]$).

Karamata $\sum_{i=1}^n \varphi(a_i) \ge \sum_{i=1}^n \varphi(b_i)$ for $a_1 \ge a_2 \ge \cdots \ge a_n$, $b_1 \ge \cdots \ge b_n$,

and $\{a_i\} \succeq \{b_i\}$ (majorization), i.e. $\sum_{i=1}^t a_i \ge \sum_{i=1}^t b_i$ for all $1 \le t \le n$,

with $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, and φ convex (for concave φ the reverse holds).

Muirhead $\sum_\pi x_{\pi(1)}^{a_1} \cdots x_{\pi(n)}^{a_n} \ge \sum_\pi x_{\pi(1)}^{b_1} \cdots x_{\pi(n)}^{b_n}$, sums over permut. π of $[n]$,

where $a_1 \ge \cdots \ge a_n$, $b_1 \ge \cdots \ge b_n$, $\{a_k\} \succeq \{b_k\}$, $x_i \ge 0$.

Hilbert $\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} \le \pi \left(\sum_{m=1}^\infty a_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty b_n^2\right)^{\frac{1}{2}}$ for $a_m, b_n \in \mathbb{R}$.

With $\max\{m,n\}$ instead of $m+n$, we have 4 instead of π.

Hardy $\sum_{n=1}^\infty \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p$ for $a_n \ge 0$, $p > 1$.

Mathieu $\frac{1}{c^2 + 1/2} < \sum_{n=1}^\infty \frac{2n}{(n^2+c^2)^2} < \frac{1}{c^2}$ for $c \ne 0$.

Kraft $\sum 2^{-c(i)} \le 1$ for $c(i)$ depth of leaf i of binary tree, sum over all leaves.

LYM $\sum\limits_{X \in \mathcal{A}} \binom{n}{|X|}^{-1} \le 1$, $\mathcal{A} \subset 2^{[n]}$, no set in $\mathcal{A}$ is subset of another set in $\mathcal{A}$.

FKG $\Pr[x \in \mathcal{A} \cap \mathcal{B}] \ge \Pr[x \in \mathcal{A}] \cdot \Pr[x \in \mathcal{B}]$, for $\mathcal{A}, \mathcal{B}$ monotone set systems.

Shearer $|\mathcal{A}|^t \le \prod_{F \in \mathcal{F}} |\mathrm{trace}_F(\mathcal{A})|$ for $\mathcal{A}, \mathcal{F} \subseteq 2^{[n]}$, where every $i \in [n]$

appears in at least t sets of $\mathcal{F}$, and $\mathrm{trace}_F(\mathcal{A}) = \{F \cap A : A \in \mathcal{A}\}$.

Sauer-Shelah $|\mathcal{A}| \le |\mathrm{str}(\mathcal{A})| \le \sum\limits_{i=0}^{\mathrm{vc}(\mathcal{A})} \binom{n}{i} \le n^{\mathrm{vc}(\mathcal{A})} + 1$ for $\mathcal{A} \subseteq 2^{[n]}$, and

$\mathrm{str}(\mathcal{A}) = \{X \subseteq [n] : \mathrm{trace}_X(\mathcal{A}) = 2^X\}$, $\mathrm{vc}(\mathcal{A}) = \max\{|X| : X \in \mathrm{str}(\mathcal{A})\}$.

Khintchine $\sqrt{\sum_i a_i^2} \ge \mathrm{E}\big[\big|\sum_i a_i r_i\big|\big] \ge \frac{1}{\sqrt{2}} \sqrt{\sum_i a_i^2}$ where $a_i \in \mathbb{R}$, and

$r_i \in \{\pm 1\}$ random variables (r.v.) i.i.d. w.pr. $\frac{1}{2}$.

Bonferroni $\Pr\big[\bigvee\limits_{i=1}^{n} A_i\big] \le \sum\limits_{j=1}^{k} (-1)^{j-1} S_j$ for $1 \le k \le n$, k odd (rev. for k even),

$S_k = \sum\limits_{1 \le i_1 < \cdots < i_k \le n} \Pr\big[A_{i_1} \wedge \cdots \wedge A_{i_k}\big]$ where A_i are events.

Bhatia-Davis $\mathrm{Var}[X] \le \big(M - \mathrm{E}[X]\big)\big(\mathrm{E}[X] - m\big)$ where $X \in [m, M]$.

Samuelson $\mu - \sigma\sqrt{n-1} \le x_i \le \mu + \sigma\sqrt{n-1}$ for $i = 1, \ldots, n$,
where $\mu = \sum x_i/n$, $\sigma^2 = \sum (x_i - \mu)^2/n$.

Markov $\Pr\big[|X| \ge a\big] \le \mathrm{E}\big[|X|\big]/a$ where X is a r.v., $a > 0$.
$\Pr\big[X \le c\big] \le (1 - \mathrm{E}[X])/(1-c)$ for $X \in [0,1]$ and $c \in \big[0, \mathrm{E}[X]\big]$.
$\Pr\big[X \in S\big] \le \mathrm{E}[f(X)]/s$ for $f \ge 0$, and $f(x) \ge s > 0$ for all $x \in S$.

Chebyshev $\Pr\big[\big|X - \mathrm{E}[X]\big| \ge t\big] \le \mathrm{Var}[X]/t^2$ where $t > 0$.
$\Pr\big[X - \mathrm{E}[X] \ge t\big] \le \mathrm{Var}[X]/(\mathrm{Var}[X] + t^2)$ where $t > 0$.

2^{nd} *moment* $\Pr\big[X > 0\big] \ge (\mathrm{E}[X])^2/(\mathrm{E}[X^2])$ where $\mathrm{E}[X] \ge 0$.
$\Pr\big[X = 0\big] \le \mathrm{Var}[X]/(\mathrm{E}[X^2])$ where $\mathrm{E}[X^2] \ne 0$.

k^{th} *moment* $\Pr\big[|X - \mu| \ge t\big] \le \dfrac{\mathrm{E}\left[(X-\mu)^k\right]}{t^k}$ and

$\Pr\big[|X - \mu| \ge t\big] \le C_k \left(\dfrac{nk}{et^2}\right)^{k/2}$ for $X_i \in [0,1]$ k-wise indep. r.v.,
$X = \sum X_i$, $i = 1, \ldots, n$, $\mu = \mathrm{E}[X]$, $C_k = 2\sqrt{\pi k}e^{1/6k}$, k even.

4^{th} *moment* $\mathrm{E}\big[|X|\big] \ge \dfrac{\left(\mathrm{E}\left[X^2\right]\right)^{3/2}}{\left(\mathrm{E}\left[X^4\right]\right)^{1/2}}$ where $0 < \mathrm{E}\left[X^4\right] < \infty$.

Chernoff $\Pr[X \ge t] \le F(a)/a^t$ for X r.v., $\Pr[X = k] = p_k$,
$F(z) = \sum_k p_k z^k$ probability gen. func., and $a \ge 1$.
$\Pr\big[X \ge \Delta\big] = \Pr\big[X \le -\Delta\big] \le \exp\left(-\Delta^2/2n\right)$ for $X_i \in \{0,1\}$ i.r.v.,
and $\Pr[X_i = \pm 1] = \frac{1}{2}$ for $i = 1, \ldots, n$, $X = \sum X_i$, $\Delta \ge 0$.

$$\Pr\big[X \ge (1+\delta)\mu\big] \le \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu} \le \exp\left(\frac{-\mu\delta^2}{2+\delta}\right)$$

$$\Pr\big[X \le (1-\delta)\mu\big] \le \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu} \le \exp\left(\frac{-\mu\delta^2}{2}\right)$$

for $X_i \in [0,1]$ i.r.v., $X = \sum X_i$, $\mu = \mathrm{E}[X]$, $\delta \ge 0$.
Further from the mean: $\Pr\big[X \ge R\big] \le 2^{-R}$ for $R \ge 2e\mu$ ($\approx 5.44\mu$).
$\Pr\big[\sum X_i \ge t\big] \le \binom{n}{k}p^k / \binom{t}{k}$ for $X_i \in \{0,1\}$ k-wise i.r.v., $\mathrm{E}[X_i] = p$.
$\Pr\big[X \ge (1+\delta)\mu\big] \le \binom{n}{\hat{k}}p^{\hat{k}} / \binom{(1+\delta)\mu}{\hat{k}}$ for $X_i \in [0,1]$ k-wise i.r.v.,
$k \ge \hat{k} = \lceil \mu\delta/(1-p) \rceil$, $\mathrm{E}[X_i] = p_i$, $X = \sum X_i$, $\mu = \mathrm{E}[X]$, $p = \frac{\mu}{n}$, $\delta > 0$.

Hoeffding $\Pr\big[\big|X - \mathrm{E}[X]\big| \ge \delta\big] \le 2\exp\left(\dfrac{-2\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$ for X_i i.r.v.,
$X_i \in [a_i, b_i]$ (w. prob. 1), $X = \sum X_i$, $\delta \ge 0$.
A related lemma, assuming $\mathrm{E}[X] = 0$, $X \in [a,b]$ (w. prob. 1) and $\lambda \in \mathbb{R}$:

$$\mathrm{E}\big[e^{\lambda X}\big] \le \exp\left(\frac{\lambda^2(b-a)^2}{8}\right)$$

Kolmogorov $\Pr\big[\max_k |S_k| \ge \varepsilon\big] \le \frac{1}{\varepsilon^2}\mathrm{Var}[S_n] = \frac{1}{\varepsilon^2}\sum_i \mathrm{Var}[X_i]$
where $X_1, \ldots, X_n$ are i.r.v., $\mathrm{E}[X_i] = 0$,
$\mathrm{Var}[X_i] < \infty$ for all i, $S_k = \sum_{i=1}^k X_i$ and $\varepsilon > 0$.

Paley-Zygmund $\Pr\big[X \ge \mu\, \mathrm{E}[X]\,\big] \ge 1 - \dfrac{\mathrm{Var}[X]}{(1-\mu)^2\,(\mathrm{E}[X])^2 + \mathrm{Var}[X]}$ for $X \ge 0$,
$\mathrm{Var}[X] < \infty$, and $\mu \in (0,1)$.

Vysochanskij-Petunin-Gauss $\Pr\big[\big|X - \mathrm{E}[X]\big| \ge \lambda\sigma\big] \le \frac{4}{9\lambda^2}$ if $\lambda \ge \sqrt{\frac{8}{3}}$,
$\Pr\big[\big|X - m\big| \ge \varepsilon\big] \le \frac{4\tau^2}{9\varepsilon^2}$ if $\varepsilon \ge \frac{2\tau}{\sqrt{3}}$,
$\Pr\big[\big|X - m\big| \ge \varepsilon\big] \le 1 - \frac{\varepsilon}{\sqrt{3}\tau}$ if $\varepsilon \le \frac{2\tau}{\sqrt{3}}$.
Where X is a unimodal r.v. with mode m,
$\sigma^2 = \mathrm{Var}[X] < \infty$, $\tau^2 = \mathrm{Var}[X] + (\mathrm{E}[X] - m)^2 = \mathrm{E}[(X-m)^2]$.

Etemadi $\Pr\big[\max_{1 \le k \le n} |S_k| \ge 3\alpha\big] \le 3 \max_{1 \le k \le n}\big(\Pr\big[\,|S_k| \ge \alpha\big]\big)$
where X_i are i.r.v., $S_k = \sum_{i=1}^k X_i$, $\alpha \ge 0$.

Doob $\Pr\big[\max_{1 \le k \le n} |X_k| \ge \varepsilon\big] \le \mathrm{E}\big[|X_n|\big]/\varepsilon$ for martingale (X_k) and $\varepsilon > 0$.

Bennett $\Pr\big[\,\sum_{i=1}^n X_i \ge \varepsilon\,\big] \le \exp\left(-\dfrac{n\sigma^2}{M^2}\,\theta\big(\dfrac{M\varepsilon}{n\sigma^2}\big)\right)$ where X_i i.r.v.,
$\mathrm{E}[X_i] = 0$, $\sigma^2 = \frac{1}{n}\sum \mathrm{Var}[X_i]$, $|X_i| \le M$ (w. prob. 1), $\varepsilon \ge 0$,
$\theta(u) = (1+u)\log(1+u) - u$.

Bernstein $\Pr\big[\,\sum_{i=1}^n X_i \ge \varepsilon\,\big] \le \exp\left(\dfrac{-\varepsilon^2}{2(n\sigma^2 + M\varepsilon/3)}\right)$ for X_i i.r.v.,
$\mathrm{E}[X_i] = 0$, $|X_i| < M$ (w. prob. 1) for all i, $\sigma^2 = \frac{1}{n}\sum \mathrm{Var}[X_i]$, $\varepsilon \ge 0$.

Azuma $\Pr\big[\big|X_n - X_0\big| \ge \delta\big] \le 2\exp\left(\dfrac{-\delta^2}{2\sum_{i=1}^n {c_i}^2}\right)$ for martingale (X_k) s.t.
$\big|X_i - X_{i-1}\big| < c_i$ (w. prob. 1), for $i = 1, \ldots, n$, $\delta \ge 0$.

Efron-Stein $\mathrm{Var}[Z] \le \frac{1}{2}\,\mathrm{E}\left[\sum_{i=1}^n \left(Z - Z^{(i)}\right)^2\right]$ for $X_i, X_i{}' \in \mathcal{X}$ i.r.v.,
$f : \mathcal{X}^n \to \mathbb{R}$, $Z = f(X_1, \ldots, X_n)$, $Z^{(i)} = f(X_1, \ldots, X_i{}', \ldots, X_n)$.

McDiarmid $\Pr\big[\big|Z - \mathrm{E}[Z]\big| \ge \delta\big] \le 2\exp\left(\dfrac{-2\delta^2}{\sum_{i=1}^n c_i^2}\right)$ for $X_i, X_i{}' \in \mathcal{X}$ i.r.v.,
Z, $Z^{(i)}$ as before, s.t. $\big|Z - Z^{(i)}\big| \le c_i$ for all i, and $\delta \ge 0$.

Janson $M \le \Pr\big[\bigwedge \overline{B}_i\big] \le M\exp\left(\dfrac{\Delta}{2-2\varepsilon}\right)$ where $\Pr[B_i] \le \varepsilon$ for all i,
$M = \prod(1 - \Pr[B_i])$, $\Delta = \sum_{i \ne j, B_i \sim B_j} \Pr[B_i \wedge B_j]$.

Lovász $\Pr\big[\bigwedge \overline{B}_i\big] \ge \prod(1 - x_i) > 0$ where $\Pr[B_i] \le x_i \cdot \prod_{(i,j) \in D}(1 - x_j)$,
for $x_i \in [0,1)$ for all $i = 1, \ldots, n$ and D the dependency graph.
If each B_i mutually indep. of all other events, except at most d,
$\Pr[B_i] \le p$ for all $i = 1, \ldots, n$, then if $ep(d+1) \le 1$ then $\Pr\big[\bigwedge \overline{B}_i\big] > 0$.
