Useful Inequalities $\{x^2 \geqslant 0\}$ v0.30b · June 8, 2018

Cauchy-Schwarz $\left(\sum_{i=1}^n x_i y_i\right)^2 \le \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right)$

Minkowski $\left(\sum_{i=1}^n |x_i+y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$ for $p \ge 1$.

Hölder $\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \left(\sum_{i=1}^n |y_i|^q\right)^{1/q}$ for $p, q > 1$, $\frac{1}{p}+\frac{1}{q}=1$.

Bernoulli

$(1+x)^r \ge 1+rx$ for $x \ge -1$, $r \in \mathbb{R} \setminus (0,1)$. Reverse for $r \in [0,1]$.

$(1+x)^r \le 1+(2^r-1)x$ for $x \in [0,1]$, $r \in \mathbb{R} \setminus (0,1)$.

$(1+x)^n \le \frac{1}{1-nx}$ for $x \in [-1,0]$, $n \in \mathbb{N}$.

$(1+x)^r \le 1+\frac{rx}{1-(r-1)x}$ for $x \in [-1, \frac{1}{r-1})$, $r > 1$.

$(1+nx)^{n+1} \ge (1+(n+1)x)^n$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$.

$x^y > \frac{x}{x+y}$ for $x > 0$, $y \in (0,1)$.

$(a+b)^n \le a^n + nb(a+b)^{n-1}$ for $a, b \ge 0$, $n \in \mathbb{N}$.

exponential

$e^x \ge \left(1+\frac{x}{n}\right)^n \ge 1+x$, $\left(1+\frac{x}{n}\right)^n \ge e^x\left(1-\frac{x^2}{n}\right)$ for $n > 1$, $|x| \le n$.

$e^x \ge x^e$ for $x \ge 0$, and $\frac{x^n}{n!}+1 \le e^x \le \left(1+\frac{x}{n}\right)^{n+x/2}$ for $x, n > 0$.

$e^x \ge 1+x+\frac{x^2}{2}$ for $x \ge 0$, reverse otherwise.

$a^x \le 1+(a-1)x$ and $a^{-x} \le 1-\frac{(a-1)}{a}x$ for $x \in [0,1]$, $a \ge 1$.

$\frac{1}{2-x} < x^x < x^2-x+1$ for $x \in (0,1]$.

$x^{1/r}(x-1) \le rx(x^{1/r}-1)$ for $x, r \ge 1$.

$x^y + y^x > 1$ and $e^x > \left(1+\frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}$ for $x, y > 0$.

$2-y-e^{-x-y} \le 1+x \le y+e^{x-y}$, and $e^x \le x+e^{x^2}$ for $x, y \in \mathbb{R}$.

$\left(1+\frac{x}{p}\right)^p \ge \left(1+\frac{x}{q}\right)^q$ for *(i)* $x > 0$, $p > q > 0$,
(ii) $-p < -q < x < 0$, *(iii)* $-q > -p > x > 0$. Reverse for:
(iv) $q < 0 < p$, $-q > x > 0$, *(v)* $q < 0 < p$, $-p < x < 0$.

logarithm

$\frac{x-1}{x} \le \ln(x) \le \frac{x^2-1}{2x} \le x-1$, $\ln(x) \le n(x^{\frac{1}{n}}-1)$ for $x, n > 0$.

$\frac{2x}{2+x} \le \ln(1+x) \le \frac{x}{\sqrt{x+1}}$ for $x \ge 0$, reverse for $x \in (-1,0]$.

$\ln(n+1) < \ln(n)+\frac{1}{n} \le \sum_{i=1}^n \frac{1}{i} \le \ln(n)+1$

$\ln(1+x) \ge \frac{x}{2}$ for $x \in [0, \sim 2.51]$, reverse elsewhere.

$\ln(1+x) \ge x-\frac{x^2}{2}+\frac{x^3}{4}$ for $x \in [0, \sim 0.45]$, reverse elsewhere.

$\ln(1-x) \ge -x-\frac{x^2}{2}-\frac{x^3}{2}$ for $x \in [0, \sim 0.43]$, reverse elsewhere.

trigonometric $x-\frac{x^3}{2} \le x\cos x \le \frac{x\cos x}{1-x^2/3} \le x\sqrt[3]{\cos x} \le x-x^3/6 \le x\cos\frac{x}{\sqrt{3}} \le \sin x$,

hyperbolic $x\cos x \le \frac{x^3}{\sinh^2 x} \le x\cos^2(x/2) \le \sin x \le (x\cos x+2x)/3 \le \frac{x^2}{\sinh x}$,

$\max\left\{\frac{2}{\pi}, \frac{\pi^2-x^2}{\pi^2+x^2}\right\} \le \frac{\sin x}{x} \le \cos\frac{x}{2} \le 1 \le 1+\frac{x^2}{3} \le \frac{\tan x}{x}$ for $x \in \left[0, \frac{\pi}{2}\right]$.

square root

$2\sqrt{x+1}-2\sqrt{x} < \frac{1}{\sqrt{x}} < \sqrt{x+1}-\sqrt{x-1} < 2\sqrt{x}-2\sqrt{x-1}$ for $x \ge 1$.

$1-\frac{x}{2}-\frac{x^2}{2} \le \sqrt{1-x} \le 1-\frac{x}{2}$.

binomial

$\max\left\{\frac{n^k}{k^k}, \frac{(n-k+1)^k}{k!}\right\} \le \binom{n}{k} \le \frac{n^k}{k!} \le \left(\frac{en}{k}\right)^k$ and $\binom{n}{k} \le \frac{n^n}{k^k(n-k)^{n-k}} \le 2^n$.

$\frac{n^k}{4k!} \le \binom{n}{k}$ for $\sqrt{n} \ge k \ge 0$ and $\frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{8n}\right) \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{9n}\right)$.

$\binom{n_1}{k_1}\binom{n_2}{k_2} \le \binom{n_1+n_2}{k_1+k_2}$ for $n_1 \ge k_1 \ge 0$, $n_2 \ge k_2 \ge 0$.

$\frac{\sqrt{\pi}}{2}G \le \binom{n}{\alpha n} \le G$ for $G = \frac{2^{nH(\alpha)}}{\sqrt{2\pi n\alpha(1-\alpha)}}$, $H(x) = -\log_2(x^x(1-x)^{1-x})$.

$\sum_{i=0}^d \binom{n}{i} \le \min\left\{n^d+1, \left(\frac{en}{d}\right)^d, 2^n\right\}$, for $n \ge d \ge 1$.

$\sum_{i=0}^{\alpha n} \binom{n}{i} \le \min\left\{\frac{1-\alpha}{1-2\alpha}\binom{n}{\alpha n}, 2^n \cdot \exp\left(-2n\left(\frac{1}{2}-\alpha\right)^2\right)\right\}$ for $\alpha \in (0, \frac{1}{2})$.

Stirling $e\left(\frac{n}{e}\right)^n \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n+1)} \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n} \le en\left(\frac{n}{e}\right)^n$

means $\min x_i \le \frac{n}{\sum x_i^{-1}} \le \left(\prod x_i\right)^{1/n} \le \frac{1}{n}\sum x_i \le \sqrt{\frac{1}{n}\sum {x_i}^2} \le \frac{\sum x_i^2}{\sum x_i} \le \max x_i$

power means $M_p \le M_q$ for $p \le q$, where $M_p = \left(\sum_i w_i|x_i|^p\right)^{1/p}$, $w_i \ge 0$, $\sum_i w_i = 1$.
In the limit $M_0 = \prod_i |x_i|^{w_i}$, $M_{-\infty} = \min_i\{x_i\}$, $M_\infty = \max_i\{x_i\}$.

Lehmer $\frac{\sum_i w_i|x_i|^p}{\sum_i w_i|x_i|^{p-1}} \le \frac{\sum_i w_i|x_i|^q}{\sum_i w_i|x_i|^{q-1}}$ for $p \le q$, $w_i \ge 0$.

log mean $\sqrt{xy} \le \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)(xy)^{\frac{1}{4}} \le \frac{x-y}{\ln(x)-\ln(y)} \le \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \le \frac{x+y}{2}$ for $x, y > 0$.

Heinz $\sqrt{xy} \le \frac{x^{1-\alpha}y^\alpha + x^\alpha y^{1-\alpha}}{2} \le \frac{x+y}{2}$ for $x, y > 0$, $\alpha \in [0,1]$.

Maclaurin-Newton ${S_k}^2 \ge S_{k-1}S_{k+1}$ and $\sqrt[k]{S_k} \ge \sqrt[(k+1)]{S_{k+1}}$ for $1 \le k < n$,
$S_k = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1}a_{i_2}\cdots a_{i_k}$, and $a_i \ge 0$.

Jensen $\varphi\left(\sum_i p_i x_i\right) \le \sum_i p_i \varphi(x_i)$ where $p_i \ge 0$, $\sum p_i = 1$, and φ convex.
Alternatively: $\varphi(\mathrm{E}[X]) \le \mathrm{E}[\varphi(X)]$. For concave φ the reverse holds.

Chebyshev $\sum_{i=1}^n f(a_i)g(b_i)p_i \ge \left(\sum_{i=1}^n f(a_i)p_i\right)\left(\sum_{i=1}^n g(b_i)p_i\right) \ge \sum_{i=1}^n f(a_i)g(b_{n-i+1})p_i$
for $a_1 \le \cdots \le a_n$, $b_1 \le \cdots \le b_n$ and f, g nondecreasing, $p_i \ge 0$, $\sum p_i = 1$.
Alternatively: $\mathrm{E}\left[f(X)g(X)\right] \ge \mathrm{E}\left[f(X)\right]\mathrm{E}\left[g(X)\right]$.

rearrangement $\sum_{i=1}^n a_i b_i \ge \sum_{i=1}^n a_i b_{\pi(i)} \ge \sum_{i=1}^n a_i b_{n-i+1}$ for $a_1 \le \cdots \le a_n$,
$b_1 \le \cdots \le b_n$ and π a permutation of $[n]$. More generally:
$\sum_{i=1}^n f_i(b_i) \ge \sum_{i=1}^n f_i(b_{\pi(i)}) \ge \sum_{i=1}^n f_i(b_{n-i+1})$
with $\left(f_{i+1}(x)-f_i(x)\right)$ nondecreasing for all $1 \le i < n$.

Weierstrass $\prod_i (1-x_i)^{w_i} \geq 1 - \sum_i w_i x_i$ where $x_i \leq 1$, and either $w_i \geq 1$ (for all i) or $w_i \leq 0$ (for all i).
If $w_i \in [0,1]$, $\sum w_i \leq 1$ and $x_i \leq 1$, the reverse holds.

Kantorovich $\left(\sum_i x_i{}^2\right)\left(\sum_i y_i{}^2\right) \leq \left(\frac{A}{G}\right)^2 \left(\sum_i x_i y_i\right)^2$ for $x_i, y_i > 0$,
$0 < m \leq \frac{x_i}{y_i} \leq M < \infty$, $A = (m+M)/2$, $G = \sqrt{mM}$.

sum & integral $\int_{L-1}^{U} f(x)\,dx \leq \sum_{i=L}^{U} f(i) \leq \int_{L}^{U+1} f(x)\,dx$ for f nondecreasing.

Cauchy $\varphi'(a) \leq \frac{f(b)-f(a)}{b-a} \leq \varphi'(b)$ where $a < b$, and φ convex.

Hermite $\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(x)\,dx \leq \frac{\varphi(a)+\varphi(b)}{2}$ for φ convex.

Gibbs $\sum_i a_i \log \frac{a_i}{b_i} \geq a \log \frac{a}{b}$ for $a_i, b_i \geq 0$, or more generally:
$\sum_i a_i \varphi\big(\frac{b_i}{a_i}\big) \leq a\,\varphi\big(\frac{b}{a}\big)$ for φ concave, and $a = \sum a_i$, $b = \sum b_i$.

Chong $\sum\limits_{i=1}^{n} \frac{a_i}{a_{\pi(i)}} \geq n$ and $\prod\limits_{i=1}^{n} a_i{}^{a_i} \geq \prod\limits_{i=1}^{n} a_i{}^{a_{\pi(i)}}$ for $a_i > 0$.

Schur $x^t(x-y)^k(x-z)^k + y^t(y-z)^k(y-x)^k + z^t(z-x)^k(z-y)^k \geq 0$
where $x, y, z, t, k \geq 0$.

Young $\left(\frac{1}{px^p} + \frac{1}{qy^q}\right)^{-1} \leq xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ for $x, y \geq 0$, $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$.

Shapiro $\sum\limits_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2}$ where $x_i > 0$, $(x_{n+1}, x_{n+2}) := (x_1, x_2)$,
and $n \leq 12$ if even, $n \leq 23$ if odd.

Hadamard $(\det A)^2 \leq \prod\limits_{i=1}^{n} \sum\limits_{j=1}^{n} A_{ij}^2$ where A is an $n \times n$ matrix.

Schur $\sum_{i=1}^n \lambda_i^2 \leq \sum_{i,j=1}^n A_{ij}^2$ and $\sum_{i=1}^k d_i \leq \sum_{i=1}^k \lambda_i$ for $1 \leq k \leq n$.
A is an $n \times n$ matrix. For the second inequality A is symmetric.
$\lambda_1 \geq \cdots \geq \lambda_n$ the eigenvalues, $d_1 \geq \cdots \geq d_n$ the diagonal elements.

Ky Fan $\frac{\prod_{i=1}^n x_i^{a_i}}{\prod_{i=1}^n (1-x_i)^{a_i}} \leq \frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i(1-x_i)}$ for $x_i \in [0, \frac{1}{2}]$, $a_i \in [0,1]$, $\sum a_i = 1$.

Aczél $\left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2 \geq \left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right)$
given that $a_1^2 > \sum_{i=2}^n a_i^2$ or $b_1^2 > \sum_{i=2}^n b_i^2$.

Mahler $\prod\limits_{i=1}^{n} (x_i + y_i)^{1/n} \geq \prod\limits_{i=1}^{n} x_i^{1/n} + \prod\limits_{i=1}^{n} y_i^{1/n}$ where $x_i, y_i > 0$.

Abel $b_1 \min\limits_k \sum\limits_{i=1}^{k} a_i \leq \sum\limits_{i=1}^{n} a_i b_i \leq b_1 \max\limits_k \sum\limits_{i=1}^{k} a_i$ for $b_1 \geq \cdots \geq b_n \geq 0$.

Milne $\left(\sum_{i=1}^n (a_i + b_i)\right)\left(\sum_{i=1}^n \frac{a_i b_i}{a_i + b_i}\right) \leq \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right)$

Carleman $\sum_{k=1}^n \left(\prod_{i=1}^k |a_i|\right)^{1/k} \leq e \sum_{k=1}^n |a_k|$

sum & product $\sum\limits_{j=1}^{m} \prod\limits_{i=1}^{n} a_{ij} \geq \sum\limits_{j=1}^{m} \prod\limits_{i=1}^{n} a_{i\pi(j)}$ and $\prod\limits_{j=1}^{m} \sum\limits_{i=1}^{n} a_{ij} \leq \prod\limits_{j=1}^{m} \sum\limits_{i=1}^{n} a_{i\pi(j)}$
where $0 \leq a_{i1} \leq \cdots \leq a_{im}$ for $i = 1, \ldots, n$ and π is a permutation of $[n]$.

$\left|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i\right| \leq \sum_{i=1}^n |a_i - b_i|$ for $|a_i|, |b_i| \leq 1$.

$\prod_{i=1}^n (\alpha + a_i) \geq (1+\alpha)^n$, where $\prod_{i=1}^n a_i \geq 1$, $a_i > 0$, $\alpha > 0$.

Callebaut $\left(\sum_i a_i^{1+x} b_i^{1-x}\right)\left(\sum_i a_i^{1-x} b_i^{1+x}\right) \geq \left(\sum_i a_i^{1+y} b_i^{1-y}\right)\left(\sum_i a_i^{1-y} b_i^{1+y}\right)$
for $1 \geq x \geq y \geq 0$, and $i = 1, \ldots, n$.

Karamata $\sum_{i=1}^n \varphi(a_i) \geq \sum_{i=1}^n \varphi(b_i)$ for $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$,
and $\{a_i\} \succeq \{b_i\}$ (majorization), i.e. $\sum_{i=1}^t a_i \geq \sum_{i=1}^t b_i$ for all $1 \leq t \leq n$,
with equality for $t = n$ and φ is convex (for concave φ the reverse holds).

Muirhead $\frac{1}{n!}\sum_\pi x_{\pi(1)}^{a_1} \cdots x_{\pi(n)}^{a_n} \geq \frac{1}{n!}\sum_\pi x_{\pi(1)}^{b_1} \cdots x_{\pi(n)}^{b_n}$
where $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ and $\{a_k\} \succeq \{b_k\}$,
$x_i \geq 0$ and the sums extend over all permutations π of $[n]$.

Hilbert $\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} \leq \pi \left(\sum_{m=1}^\infty a_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty b_n^2\right)^{\frac{1}{2}}$ for $a_m, b_n \in \mathbb{R}$.
With $\max\{m, n\}$ instead of $m + n$, we have 4 instead of π.

Hardy $\sum_{n=1}^\infty \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p$ for $a_n \geq 0$, $p > 1$.

Carlson $\left(\sum_{n=1}^\infty a_n\right)^4 \leq \pi^2 \sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty n^2 a_n^2$ for $a_n \in \mathbb{R}$.

Mathieu $\frac{1}{c^2 + 1/2} < \sum_{n=1}^\infty \frac{2n}{(n^2+c^2)^2} < \frac{1}{c^2}$ for $c \neq 0$.

LYM $\sum\limits_{X \in \mathcal{A}} \binom{n}{|X|}^{-1} \leq 1$, $\mathcal{A} \subset 2^{[n]}$, no set in $\mathcal{A}$ is subset of another set in $\mathcal{A}$.

Sauer-Shelah $|\mathcal{A}| \leq |\mathrm{str}(\mathcal{A})| \leq \sum\limits_{i=0}^{\mathrm{vc}(\mathcal{A})} \binom{n}{i}$ for $\mathcal{A} \subseteq 2^{[n]}$, and
$\mathrm{str}(\mathcal{A}) = \{X \subseteq [n] : X \text{ shattered by } \mathcal{A}\}$, $\mathrm{vc}(\mathcal{A}) = \max\{|X| : X \in \mathrm{str}(\mathcal{A})\}$.

Bonferroni $\Pr\big[\bigvee\limits_{i=1}^{n} A_i\big] \leq \sum\limits_{j=1}^{k} (-1)^{j-1} S_j$ for $1 \leq k \leq n$, k odd,

$\Pr\big[\bigvee\limits_{i=1}^{n} A_i\big] \geq \sum\limits_{j=1}^{k} (-1)^{j-1} S_j$ for $2 \leq k \leq n$, k even.

$S_k = \sum\limits_{1 \leq i_1 < \cdots < i_k \leq n} \Pr\big[A_{i_1} \wedge \cdots \wedge A_{i_k}\big]$ where A_i are events.

Bhatia-Davis $\mathrm{Var}[X] \leq \big(M - \mathrm{E}[X]\big)\big(\mathrm{E}[X] - m\big)$ where $X \in [m, M]$.

Samuelson $\mu - \sigma\sqrt{n-1} \leq x_i \leq \mu + \sigma\sqrt{n-1}$ for $i = 1, \ldots, n$.
Where $\mu = \sum x_i/n$, $\sigma^2 = \sum (x_i - \mu)^2/n$.

Markov $\Pr\big[|X| \geq a\big] \leq \mathrm{E}\big[|X|\big]/a$ where X is a random variable (r.v.), $a > 0$.

$\Pr\big[X \leq c\big] \leq (1-\mathrm{E}[X])/(1-c)$ for $X \in [0,1]$ and $c \in \big[0, \mathrm{E}[X]\big]$.

$\Pr\big[X \in S\big] \leq \mathrm{E}[f(X)]/s$ for $f \geq 0$, and $f(x) \geq s > 0$ for all $x \in S$.

Chebyshev $\Pr\big[|X - \mathrm{E}[X]| \geq t\big] \leq \mathrm{Var}[X]/t^2$ where $t > 0$.

$\Pr\big[X - \mathrm{E}[X] \geq t\big] \leq \mathrm{Var}[X]/(\mathrm{Var}[X] + t^2)$ where $t > 0$.

2^{nd} ***moment*** $\Pr\big[X > 0\big] \geq (\mathrm{E}[X])^2/(\mathrm{E}[X^2])$ where $\mathrm{E}[X] \geq 0$.

$\Pr\big[X = 0\big] \leq \mathrm{Var}[X]/(\mathrm{E}[X^2])$ where $\mathrm{E}[X^2] \neq 0$.

k^{th} ***moment*** $\Pr\big[|X - \mu| \geq t\big] \leq \dfrac{\mathrm{E}\left[(X-\mu)^k\right]}{t^k}$ and

$\Pr\big[|X - \mu| \geq t\big] \leq C_k \left(\dfrac{nk}{t^2}\right)^{k/2}$ for $X_i \in [0,1]$ k-wise indep. r.v.,

$X = \sum X_i$, $i = 1, \ldots, n$, $\mu = \mathrm{E}[X]$, $C_k = 2\sqrt{\pi k} e^{1/6k} \leq 1.0004$, k even.

2^{nd} ***and*** 4^{th} $\mathrm{E}\big[|X|\big] \geq \dfrac{\left(\mathrm{E}\left[X^2\right]\right)^{3/2}}{\left(\mathrm{E}\left[X^4\right]\right)^{1/2}}$ where $0 < \mathrm{E}\left[X^4\right] < \infty$.

$\Pr\big[X \geq \dfrac{\sigma}{2\sqrt{t}}\big] > 0$ where $\mathrm{E}[X] = 0$, $\mathrm{E}[X^2] = \sigma^2$, $0 < \mathrm{E}[X^4] \leq t\sigma^4$.

Chernoff $\Pr[X \geq t] \leq F(a)/a^t$ for X r.v., $\Pr[X = k] = p_k$,

$F(z) = \sum_k p_k z^k$ probability gen. func., and $a \geq 1$.

$\Pr\big[X \geq (1+\delta)\mu\big] \leq \left(\dfrac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu \leq \exp\left(\dfrac{-\mu\delta^2}{3}\right)$

for X_i i.r.v. from [0,1], $X = \sum X_i$, $\mu = \mathrm{E}[X]$, $\delta \geq 0$ resp. $\delta \in [0,1)$.

$\Pr\big[X \leq (1-\delta)\mu\big] \leq \left(\dfrac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^\mu \leq \exp\left(\dfrac{-\mu\delta^2}{2}\right)$ for $\delta \in [0,1)$.

Further from the mean: $\Pr\big[X \geq R\big] \leq 2^{-R}$ for $R \geq 2e\mu$ ($\approx 5.44\mu$).

$\Pr\big[X \geq t\big] \leq \dfrac{\binom{n}{k} p^k}{\binom{t}{k}}$ for $X_i \in \{0,1\}$ k-wise i.r.v., $\mathrm{E}[X_i] = p$, $X = \sum X_i$.

$\Pr\big[X \geq (1+\delta)\mu\big] \leq \binom{n}{\hat{k}} p^{\hat{k}} / \binom{(1+\delta)\mu}{\hat{k}}$ for $X_i \in [0,1]$ k-wise i.r.v.,

$k \geq \hat{k} = \lceil \mu\delta/(1-p) \rceil$, $\mathrm{E}[X_i] = p_i$, $X = \sum X_i$, $\mu = \mathrm{E}[X]$, $p = \frac{\mu}{n}$, $\delta > 0$.

Hoeffding $\Pr\big[|X - \mathrm{E}[X]| \geq \delta\big] \leq 2\exp\left(\dfrac{-2\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$ for X_i i.r.v.,

$X_i \in [a_i, b_i]$ (w. prob. 1), $X = \sum X_i$, $\delta \geq 0$.

A related lemma, assuming $\mathrm{E}[X] = 0$, $X \in [a,b]$ (w. prob. 1) and $\lambda \in \mathbb{R}$:

$\mathrm{E}\big[e^{\lambda X}\big] \leq \exp\left(\dfrac{\lambda^2 (b-a)^2}{8}\right)$

Kolmogorov $\Pr\big[\max_k |S_k| \geq \varepsilon\big] \leq \frac{1}{\varepsilon^2} \mathrm{Var}[S_n] = \frac{1}{\varepsilon^2} \sum_i \mathrm{Var}[X_i]$

where $X_1, \ldots, X_n$ are i.r.v., $\mathrm{E}[X_i] = 0$,

$\mathrm{Var}[X_i] < \infty$ for all i, $S_k = \sum_{i=1}^k X_i$ and $\varepsilon > 0$.

Paley-Zygmund $\Pr\big[X \geq \mu\, \mathrm{E}[X]\big] \geq 1 - \dfrac{\mathrm{Var}[X]}{(1-\mu)^2\, (\mathrm{E}[X])^2 + \mathrm{Var}[X]}$ for $X \geq 0$,

$\mathrm{Var}[X] < \infty$, and $\mu \in (0,1)$.

Vysochanskij-Petunin-Gauss $\Pr\big[|X - \mathrm{E}[X]| \geq \lambda\sigma\big] \leq \frac{4}{9\lambda^2}$ if $\lambda \geq \sqrt{\frac{8}{3}}$,

$\Pr\big[|X - m| \geq \varepsilon\big] \leq \frac{4\tau^2}{9\varepsilon^2}$ if $\varepsilon \geq \frac{2\tau}{\sqrt{3}}$,

$\Pr\big[|X - m| \geq \varepsilon\big] \leq 1 - \frac{\varepsilon}{\sqrt{3}\tau}$ if $\varepsilon \leq \frac{2\tau}{\sqrt{3}}$.

Where X is a unimodal r.v. with mode m,

$\sigma^2 = \mathrm{Var}[X] < \infty$, $\tau^2 = \mathrm{Var}[X] + (\mathrm{E}[X] - m)^2 = \mathrm{E}[(X-m)^2]$.

Etemadi $\Pr\big[\max_{1 \leq k \leq n} |S_k| \geq 3\alpha\big] \leq 3 \max_{1 \leq k \leq n} \left(\Pr\big[\, |S_k| \geq \alpha\big]\right)$

where X_i are i.r.v., $S_k = \sum_{i=1}^k X_i$, $\alpha \geq 0$.

Doob $\Pr\big[\max_{1 \leq k \leq n} |X_k| \geq \varepsilon\big] \leq \mathrm{E}\big[|X_n|\big]/\varepsilon$ for martingale (X_k) and $\varepsilon > 0$.

Bennett $\Pr\big[\sum_{i=1}^n X_i \geq \varepsilon\big] \leq \exp\left(-\dfrac{n\sigma^2}{M^2}\, \theta\left(\dfrac{M\varepsilon}{n\sigma^2}\right)\right)$ where X_i i.r.v.,

$\mathrm{E}[X_i] = 0$, $\sigma^2 = \frac{1}{n}\sum \mathrm{Var}[X_i]$, $|X_i| \leq M$ (w. prob. 1), $\varepsilon \geq 0$,

$\theta(u) = (1+u)\log(1+u) - u$.

Bernstein $\Pr\big[\sum_{i=1}^n X_i \geq \varepsilon\big] \leq \exp\left(\dfrac{-\varepsilon^2}{2(n\sigma^2 + M\varepsilon/3)}\right)$ for X_i i.r.v.,

$\mathrm{E}[X_i] = 0$, $|X_i| < M$ (w. prob. 1) for all i, $\sigma^2 = \frac{1}{n}\sum \mathrm{Var}[X_i]$, $\varepsilon \geq 0$.

Azuma $\Pr\big[|X_n - X_0| \geq \delta\big] \leq 2\exp\left(\dfrac{-\delta^2}{2\sum_{i=1}^n {c_i}^2}\right)$ for martingale (X_k) s.t.

$\big|X_i - X_{i-1}\big| < c_i$ (w. prob. 1), for $i = 1, \ldots, n$, $\delta \geq 0$.

Efron-Stein $\mathrm{Var}[Z] \leq \frac{1}{2}\, \mathrm{E}\left[\sum_{i=1}^n \left(Z - Z^{(i)}\right)^2\right]$ for $X_i, X_i{}' \in \mathcal{X}$ i.r.v.,

$f : \mathcal{X}^n \to \mathbb{R}$, $Z = f(X_1, \ldots, X_n)$, $Z^{(i)} = f(X_1, \ldots, X_i{}', \ldots, X_n)$.

McDiarmid $\Pr\big[|Z - \mathrm{E}[Z]| \geq \delta\big] \leq 2\exp\left(\dfrac{-2\delta^2}{\sum_{i=1}^n c_i^2}\right)$ for $X_i, X_i{}' \in \mathcal{X}$ i.r.v.,

Z, $Z^{(i)}$ as before, s.t. $\big|Z - Z^{(i)}\big| \leq c_i$ for all i, and $\delta \geq 0$.

Janson $M \leq \Pr\big[\bigwedge \overline{B}_i\big] \leq M\exp\left(\dfrac{\Delta}{2 - 2\varepsilon}\right)$ where $\Pr[B_i] \leq \varepsilon$ for all i,

$M = \prod(1 - \Pr[B_i])$, $\Delta = \sum_{i \neq j, B_i \sim B_j} \Pr[B_i \wedge B_j]$.

Lovász $\Pr\big[\bigwedge \overline{B}_i\big] \geq \prod(1 - x_i) > 0$ where $\Pr[B_i] \leq x_i \cdot \prod_{(i,j) \in D}(1 - x_j)$,

for $x_i \in [0,1)$ for all $i = 1, \ldots, n$ and D the dependency graph.

If each B_i mutually indep. of all other events, exc. at most d,

$\Pr[B_i] \leq p$ for all $i = 1, \ldots, n$, then if $ep(d+1) \leq 1$ then $\Pr\big[\bigwedge \overline{B}_i\big] > 0$.

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