Minimum Average Distance Triangulations

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Weighted, undirected graph $G = (V, E, w)$.

$d_G(u, v)$: length of the shortest $u - v$ path.

Average distance:

$$\mathcal{A}(G) = \frac{1}{\binom{|V|}{2}} \sum_{\{x,y\} \subseteq V} d_G(u, v).$$

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Total distance (H. Wiener, 1947):

$$\mathcal{W}(G) = \sum_{\{x,y\} \subseteq V} d_G(u, v).$$

Introduction

Network design (D.S. Johnson et al., 1978):

- Given $G = (V, E, w)$, find a spanning subgraph T,

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- NP-hard, even with $w = 1$, and $B = |V| - 1$ (spanning tree).

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- No budget on total weight $\implies$ T is maximal non-crossing.

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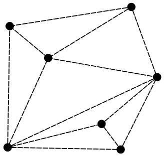

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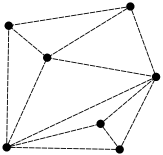

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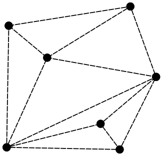

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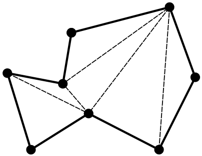
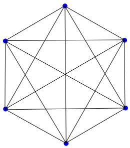


Example:
Regular hexagon, Euclidean weights.



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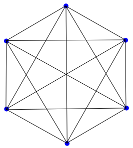

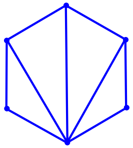

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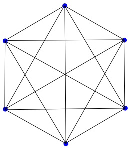
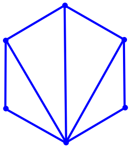


Minimum Average Distance Triangulation:
$\mathcal{W} \approx 24.93$, $\sum w(e) \approx 11.46$



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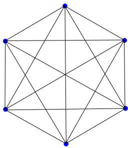

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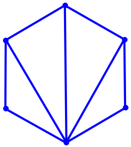

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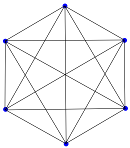

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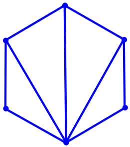

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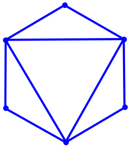

Summary of results

$$\left.\begin{array}{r}\text{point sets}\\ \text{polygons}\end{array}\right\} \left.\begin{array}{r}\text{unit weights}\\ \text{Euclidean weights}\\ \text{arbitrary weights}\end{array}\right\} \quad \min_T \sum_{x,y} d_T(x,y).$$

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Metric (e.g. Euclidean) weights: Open

1. Hardness result

MADT: point set $S \subseteq \mathbb{R}^2$, weights $w : S^2 \to \mathbb{R}$.

$$w \ \ \text{semimetric}: \ \forall x, y \in S : \begin{cases} w(x,y) \geq 0 \\ w(x,y) = 0 \ \text{ iff } \ x = y \\ w(x,y) = w(y,x) \end{cases}$$

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Decision problem: given a threshold $\mathcal{W}^\star \in \mathbb{R}$, is there a triangulation T of S, such that:

$$\mathcal{W}(T) = \sum_{x,y \in S} d_T(x,y) \leq \mathcal{W}^\star$$

MADT $\in$ **NP**: use an all-pairs-shortest-path algorithm.

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PLANAR 3SAT $\longrightarrow$ MAD TRIANGULATION

PLANAR 3SAT	MAD TRIANGULATION
• 3-CNF formula (variables, clauses) • planar embedding	• point set S • weights w • threshold $\mathcal{W}^\star$

$\exists$ satisfying assignment $\longleftrightarrow$ $\exists T$ s.t. $\mathcal{W}(T) \leq \mathcal{W}^\star$

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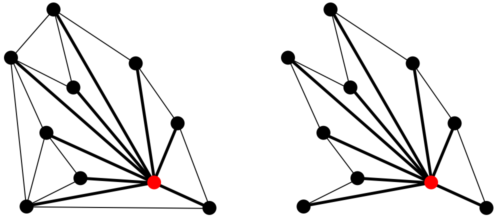

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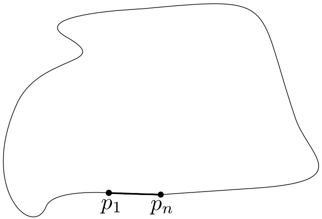

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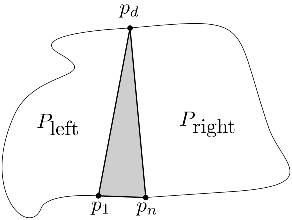

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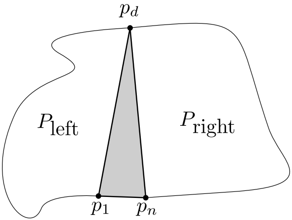

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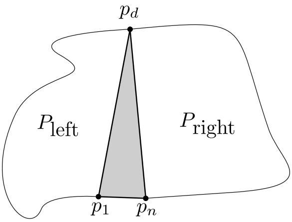


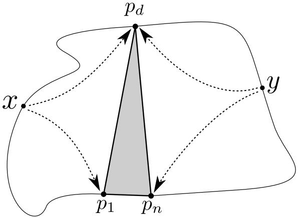
General approach: *guess* a triangle that is part of the solution, decompose and recursively solve the problem ($\rightarrow$ dynamic programming).

Problem: cost is not easy to decompose. How to deal with cross-distances?

Question: How to deal with cross-distances?

Let $x \in P_{left}$ and $y \in P_{right}$.

How does the path $x \leftrightarrow y$ cross the triangle $(p_1 p_d p_n)$?

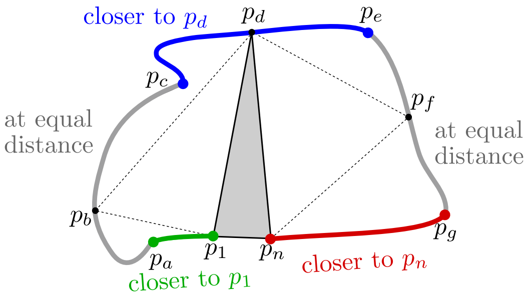


It depends on where x and y are.

Question: How to deal with cross-distances?

We assume the polygon is triangulated.

Observation: Vertices can be grouped into contiguous blocks.



$x \in P_L$ (left part), $y \in P_R$ (right part)

Depending on where x and y fall, we can tell how $x \leftrightarrow y$ crosses the triangle, so we can decompose $d_T(x, y)$:

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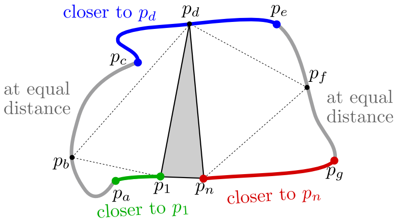

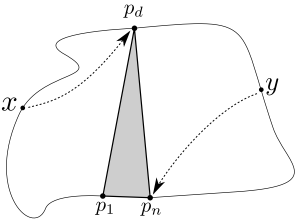

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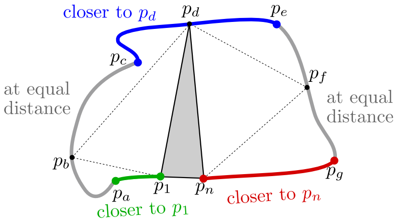
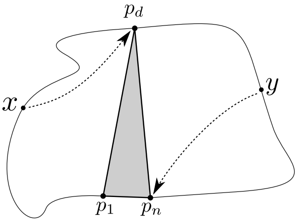
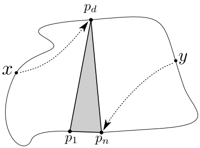




Let $\phi = d_T(x, p_d) + d_T(y, p_n)$. Then:

$$d_T(x, y) = \begin{cases} \phi - 1 & \text{if } y \in [p_d, p_e] \\ \phi + 1 & \text{if } y \in [p_g, p_n] \text{ and } x \in (p_a, p_d] \\ \phi & \text{otherwise .} \end{cases}$$



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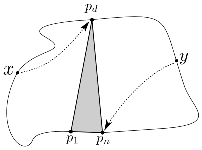

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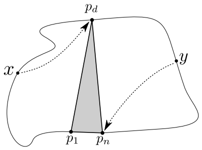

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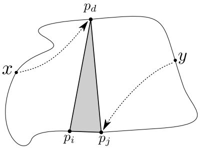

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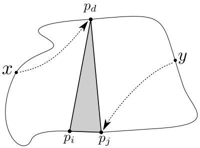

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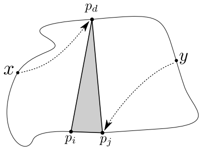

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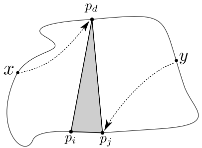
Minimizing $\mathcal{W}_{\mathrm{EXT}}(T,0)\Big|_1^n$ solves the initial problem.



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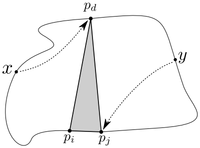
Using the formula for $d_T(x,y)$ we can express $\mathcal{W}_{\mathrm{EXT}}$ recursively:

$$\mathcal{W}_{\mathrm{EXT}}(T,\alpha)\Big|_i^j = \mathcal{W}_{\mathrm{EXT}}(T,\ \dots\ )\Big|_i^d + \mathcal{W}_{\mathrm{EXT}}(T,\ \dots\ )\Big|_d^j$$
$$+\ \dots$$



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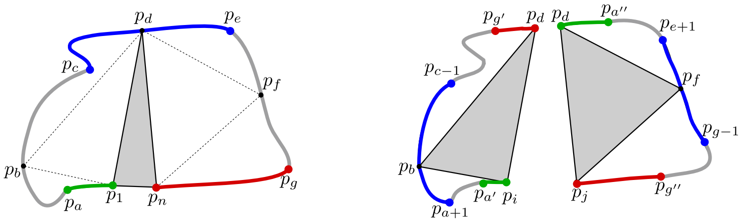
$$+ (\alpha+j-g+1)(d-a-1) + (e-d+1)(i-d).$$

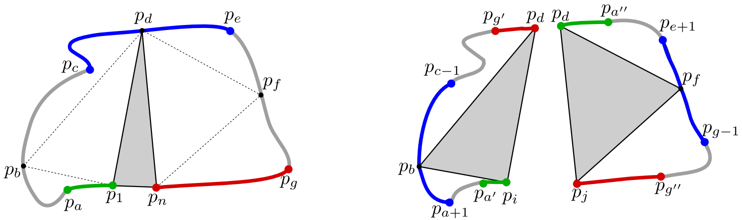
We need to ensure that the constraints on the indices are respected:



Observation: Assume T contains the triangles $p_i p_d p_j$ and $p_i p_b p_d$.

(a) a is the largest index s.t. $d_T(p_a, p_i) < d_T(p_a, p_d)$ iff $a + 1$ is the smallest index s.t. $d_T(p_{a+1}, p_b) < d_T(p_{a+1}, p_i)$.

We need to ensure that the constraints on the indices are respected:



Observation: Assume T contains the triangles $p_i p_d p_j$ and $p_i p_b p_d$.

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(b) c is the smallest index s.t. $d_T(p_c, p_d) < d_T(p_c, p_i)$ iff $c-1$ is the largest index s.t. $d_T(p_{c-1}, p_b) < d_T(p_{c-1}, p_d)$.

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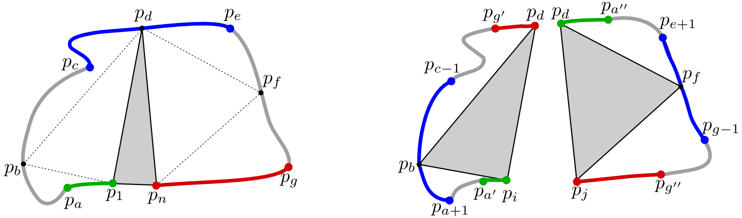

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(b) c is the smallest index s.t. $d_T(p_c, p_d) < d_T(p_c, p_i)$ iff $c - 1$ is the largest index s.t. $d_T(p_{c-1}, p_b) < d_T(p_{c-1}, p_d)$.

(c) analogous relations on the right side

Putting it all together:

procedure $EXT\left((p_i, \ldots, p_j),\ p_a, p_c, p_e, p_g,\ \alpha\right)$:

if $(a = i)$ **and** $(c = e = i + 1)$ **and** $(g = j = i + 2)$:

return $(3 + 2\alpha)$; /* *the polygon has only three vertices* */

else:

$$\textbf{return} \min_{\substack{p_d, p'_a, p'_g, p''_a, p''_g: \\ i \le a' \le a+1 \le c-1 \le g' \le d \\ d \le a'' \le e+1 \le g-1 \le g'' \le j \\ p_i \leftrightarrow p_d \leftrightarrow p_j}} \Big\{ EXT\left((p_i, \ldots, p_d),\ p'_a, p_{a+1}, p_{c-1}, p'_g,\ \alpha + j - d\right) + EXT\left((p_d, \ldots, p_j),\ p''_a, p_{e+1}, p_{g-1}, p''_g,\ \alpha + d - i\right) + (\alpha + j - g + 1)(d - a - 1) + (e - d + 1)(i - d) \Big\};$$

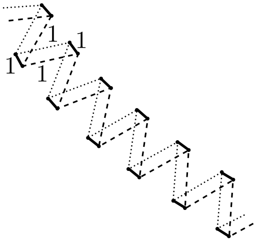
Summary of results

$$\left.\left.\begin{array}{r}\text{point sets}\\ \text{polygons}\end{array}\right\} \begin{array}{r}\text{unit weights}\\ \text{Euclidean weights}\\ \text{arbitrary weights}\end{array}\right\} \quad \min_T \sum_{x,y} d_T(x,y).$$

1. Point sets, arbitrary (semimetric) weights: NP-complete
2. Point sets, convex polygons, unit weights: trivial
3. Arbitrary simple polygons, unit weights: $\mathcal{O}(n^{11})$

Metric (e.g. Euclidean) weights: Open

Thank you for your attention.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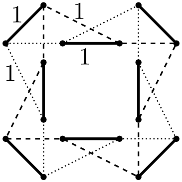
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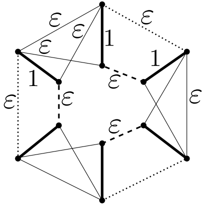
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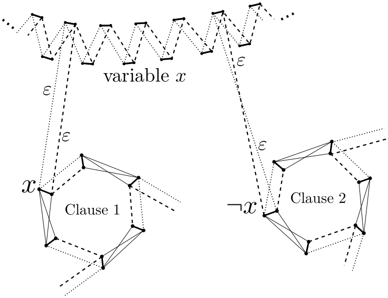
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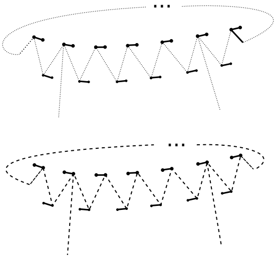
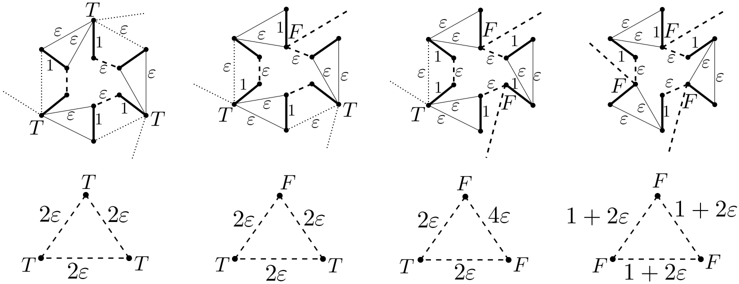
Clause 1
Clause 2

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