Multi-finger Binary Search Trees

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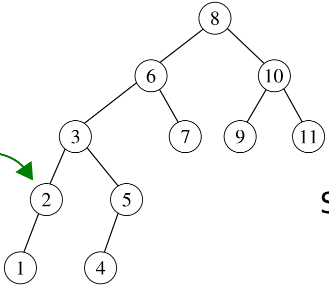
Binary Search Tree (BST)

search(2)
cost = 4



n keys

search cost = depth + 1

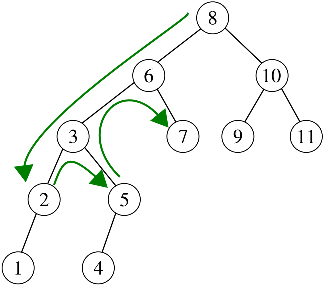
Search sequence:
$x_1, x_2, x_3, \ldots$

Tree can be adapted using rotations, to prepare for next search.
cost = pointer moves + rotations

- offline cost: ***OPT***
- online algorithms: e.g. Splay tree

[Sleator, Tarjan, 1983]

Finger Search



search(2)
search(5)
search(7)
...

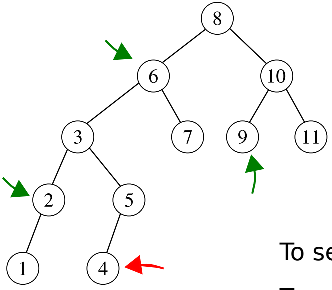
Search starts from previous location

Tree is static, chosen optimally

cost = amount of movement by finger

optimal finger search cost: ***OPT_1***

Multiple fingers



k fingers, stationed at nodes

search starts from any of the fingers

To serve request, must move some finger there

Tree is static, chosen optimally

cost = amount of movement by **all** fingers

optimal k-finger search cost: ***OPT$_k$***

Multiple fingers

$$OPT_1 \geq OPT_2 \geq \cdots \geq OPT_k \geq \cdots \geq OPT_n$$

matched by online
BST with rotations

[Cole et al., 2000]
[Iacono-Langerman, 2016]

$$OPT \leq O(OPT_1)$$

Our result

matched by online
BST with rotations
(with small overhead)

$$OPT \leq O(OPT_k) \cdot \log k$$

Our results

1. BSTs (with rotations) can simulate the k-finger optimum with small overhead:

$$OPT \leq O(OPT_k) \cdot \log k$$

log(k) factor is optimal.

2. There is an online BST with cost:

$$O(OPT_k) \cdot \log^7 k$$

The proof has three ingredients:

1. Simulate k-finger strategy by BST
   with single root-pointer (and rotations)

2. Find online k-finger strategy

3. Learn optimal underlying tree

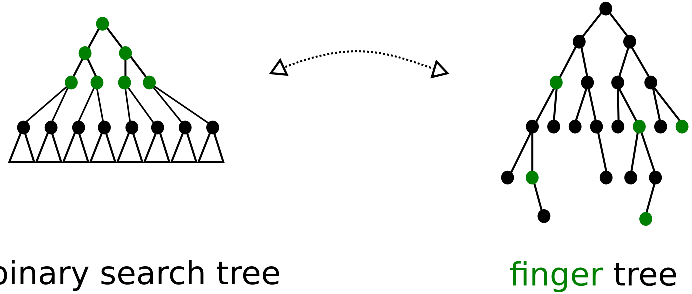
1. Simulate k-finger strategy by BST

with optimal O(log k) overhead,

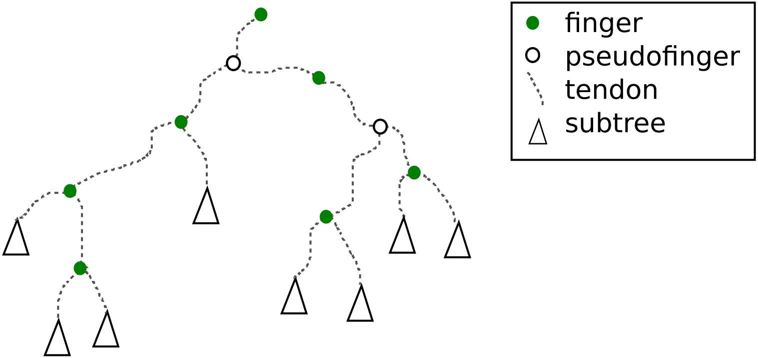
refining previous approach with overhead O(k)

[Demaine, Iacono, Langerman, Özkan, 2013]

Idea: store nodes with fingers as a balanced subtree



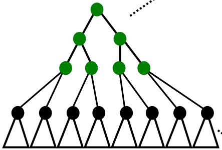
finger tree



O(k) fingers and pseudofingers,
can store as e.g. AVL-tree

tendons: almost sorted,
can update in O(1) amortized time

BST simulation of finger tree



fingers and pseudofingers
stored here e.g. AVL-tree

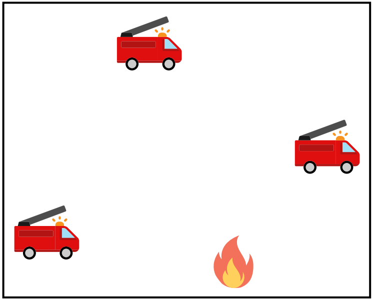
tendons and subtrees stored here
tendons stored as BST deque

access finger --> search AVL tree ... O(log k)

move finger --> update connected tendons
insert/delete in deque ... O(1)
update node role as (pseudo)finger;
may trigger AVL re-balancing ... O(log k)

2. Find online k-finger strategy

metric space




servers
requests

k-server problem

Goal: serve requests,
minimize total movement

Our task is the special case when the metric is a BST

Relevant result: $O(\log^6 k)$ competitive online algorithm
[Lee; Bubeck et al. 2018]

3. Learn best underlying tree

Search sequence:

$x_1, x_2, x_3, \ldots$

- **Idea:** Process in epochs of length O(nlogn)
- Maintain distribution over all ~4^n trees

When epoch starts:

pick a tree from distribution, rotate to it

(rotation cost amortized over epoch)

When epoch ends:

evaluate all trees, update distribution using

multiplicative-weights-update (MWU)

Loss of tree T = cost during epoch if we used T

MWU guarantee

nr. of possible trees ~ 4^n

max possible cost within epoch ~ n^2

overall cost < $(1+\varepsilon)T^* + \dfrac{\ln N \cdot T^{max}}{\varepsilon}$

total cost with best tree in hindsight

poly(n) additive term

Summary of main result

1. Simulate k-finger strategy by BST
   with single root-pointer (and rotations)

   O(logk) factor loss

   refines technique of [Demaine et al.]

2. Find online k-finger strategy

   $O(\log^6 k)$ factor loss

   first connection between k-server and BST?

3. Learn optimal underlying tree

   O(1+eps) factor, additive term

   MWU technique, randomized

Applications

1. Better understand limitations of BST model

[Sleator, Tarjan, 1983]:
Splay tree cost is O(OPT)

Dynamic Optimality Conjecture

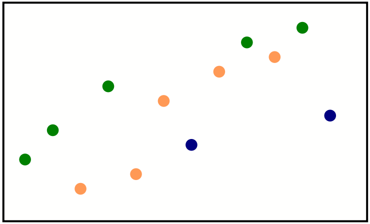
From our results:
then it must be $O(OPT_k)$ logk

(even k=1 case is hard to prove)

barrier to Dynamic Optimality

Applications

2. Understand which search sequences are easy



Example:
k-monotone sequence
(shuffle of k runs)

k-finger strategy:
each finger serves one run
($O(n)$ traversal of tree)
total cost: $O(nk)$

=> BST cost: $O(n\ k\log k)$

Applications

2. Understand which search sequences are easy

Example:
weak unified bound
(search is close to a recent search)

amortized cost for search(x_k):

$$\min_t\{\log|x_k - x_{k-t}|\} \cdot f(t)$$

rank-diff. from recent search

penalty factor for looking back too far

we give a k-finger strategy (not so easy)

OPEN QUESTIONS

1. De-randomize our online BST

(randomness comes from picking tree in MWU strategy)

Possible randomized/deterministic separation?

2. Towards dynamic optimality

Show that Splay tree matches $O(OPT_k)$ f(k)

Show that some online BST matches $O(OPT_k)$ logk

(another log(k) factor unavoidable if online k-server is used)