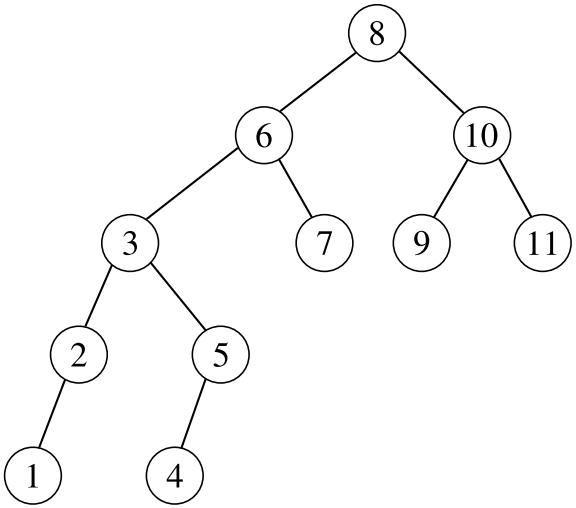
Self-Adjusting BSTs
What Makes Them Tick?

Parinya Chalermsook[1]
Mayank Goswami[1]
László Kozma[2]
Kurt Mehlhorn[1]
Thatchaphol Saranurak[3]

[1]Max-Planck Institute for Informatics
[2]Department of Computer Science, Saarland University
[3]KTH Royal Institute of Technology, Stockholm.

Introduction: Binary Search Trees



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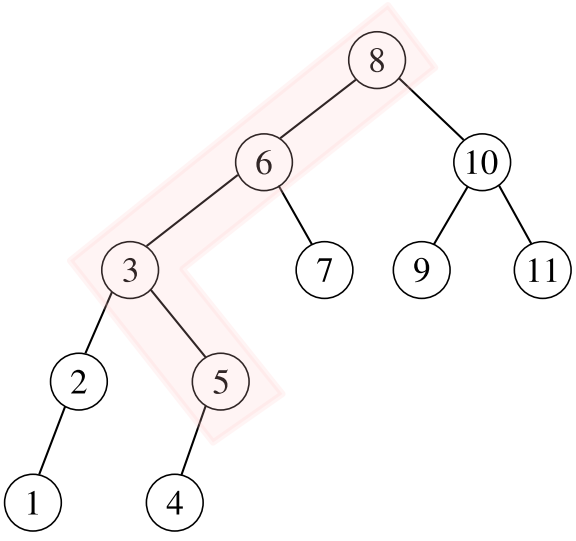

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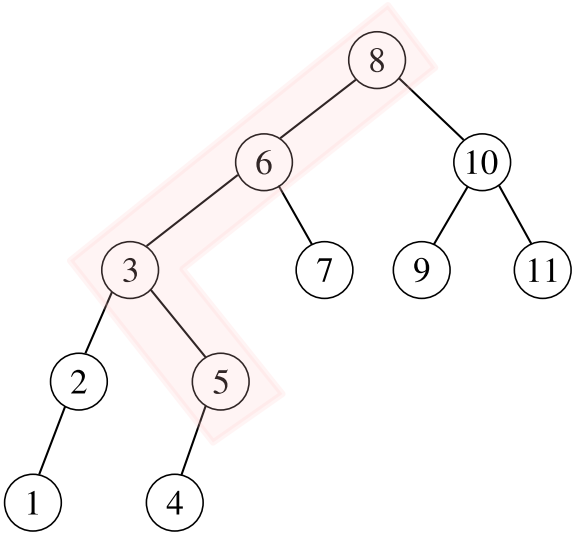

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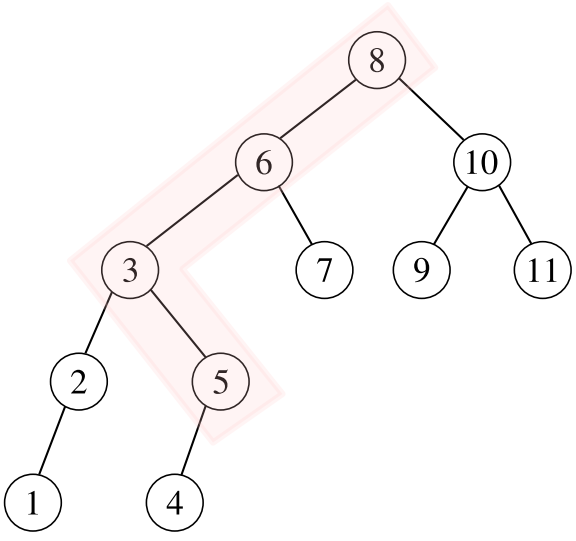

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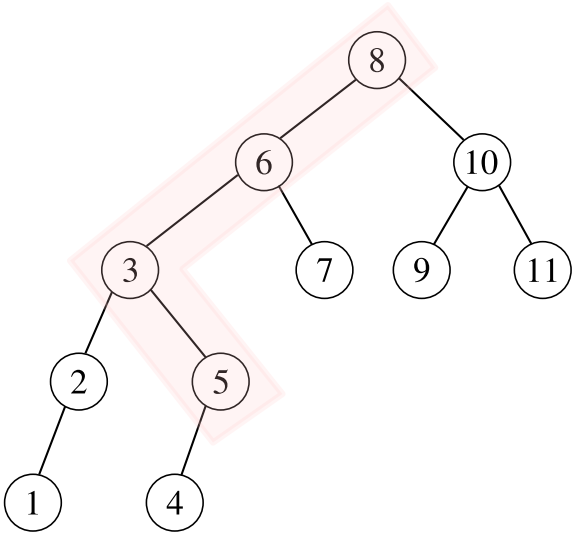

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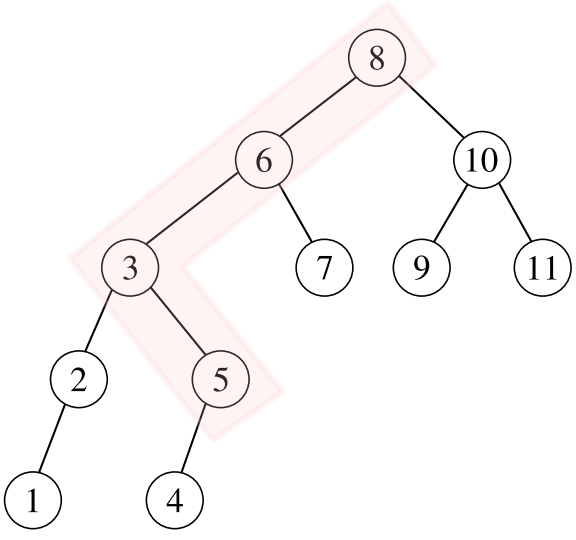

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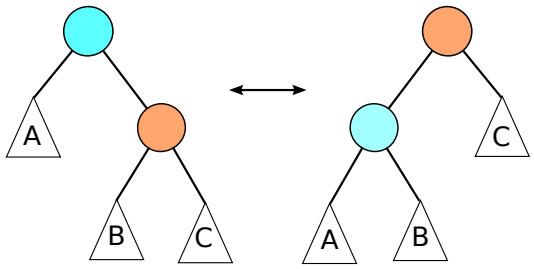
Optimal tree for a given sequence
(Knuth, 1971)

Introduction: Self-Adjusting BST

What if we can re-arrange the tree between accesses?



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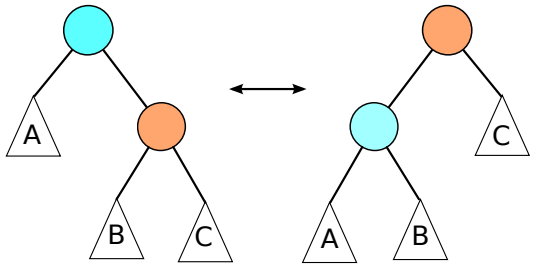

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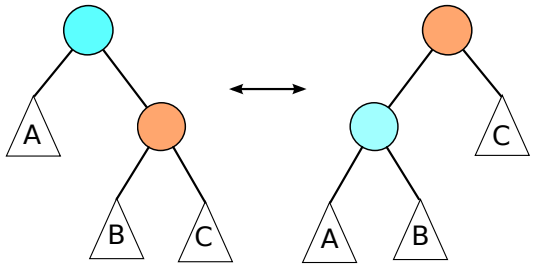

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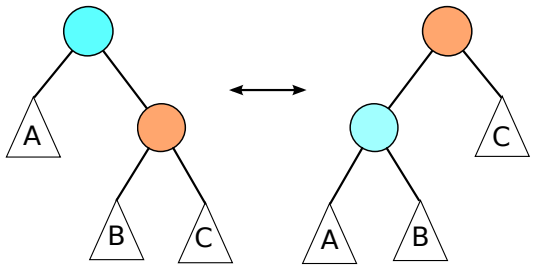

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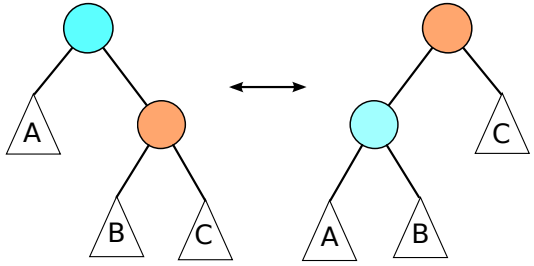

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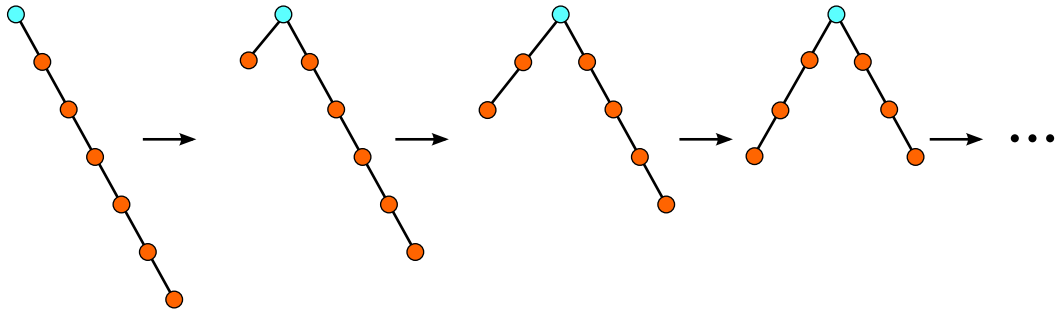

Splay Trees (Sleator and Tarjan, 1983)

After each access:
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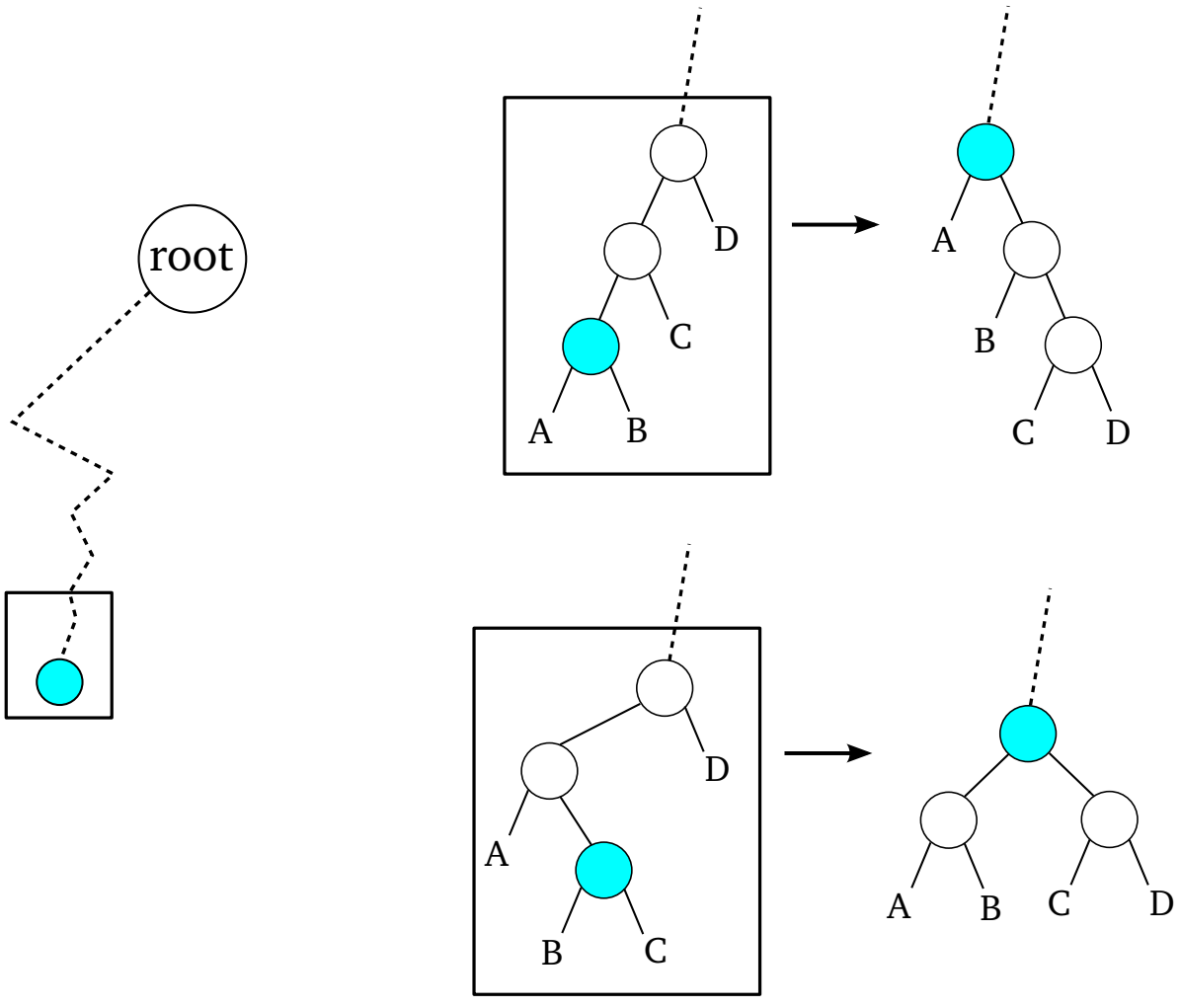

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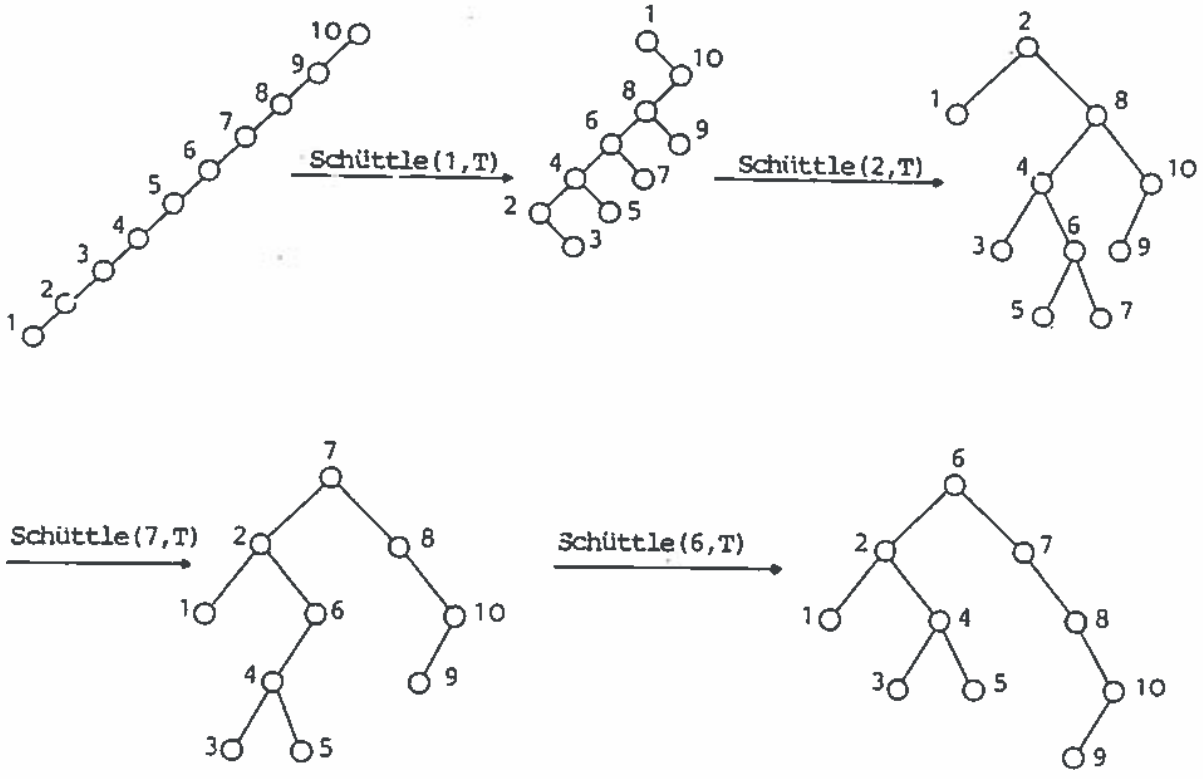

Abb. 95. 4 Schüttle-Operationen.

[K. Mehlhorn: Data structures and algorithms (German ed. 1986)]

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First 4 properties captured by **access lemma**.

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Access Lemma for Splay Trees (simplified, unweighted)

Let P be the path to the accessed element s in T, and let T' be the tree after splaying s. Then

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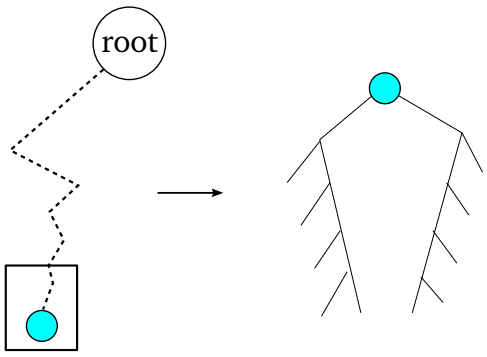

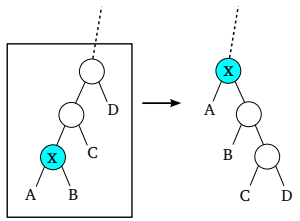

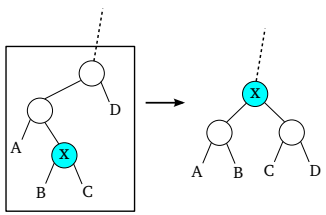

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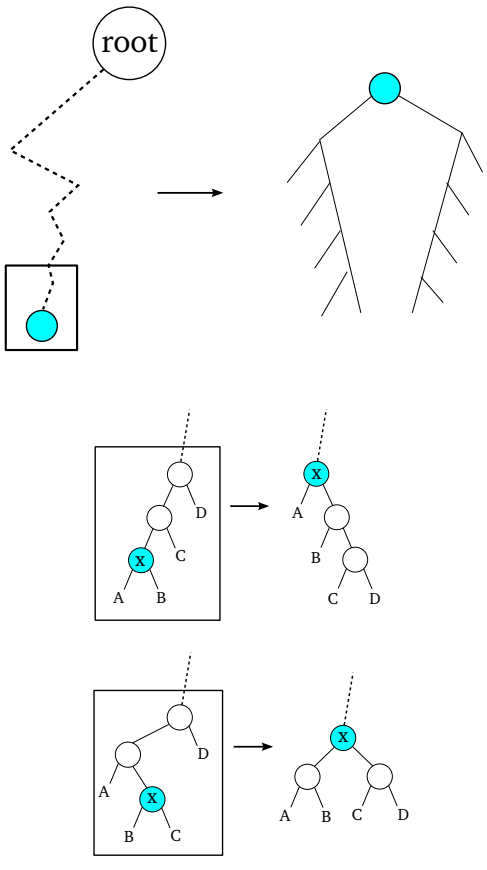

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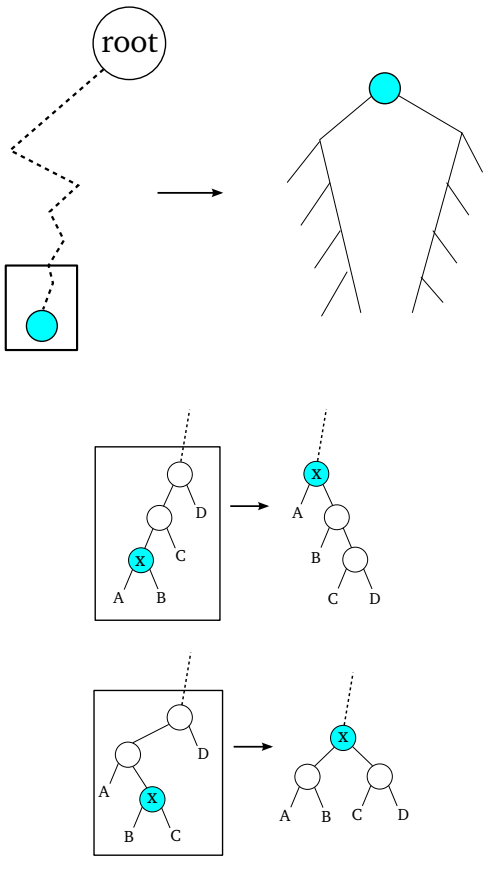

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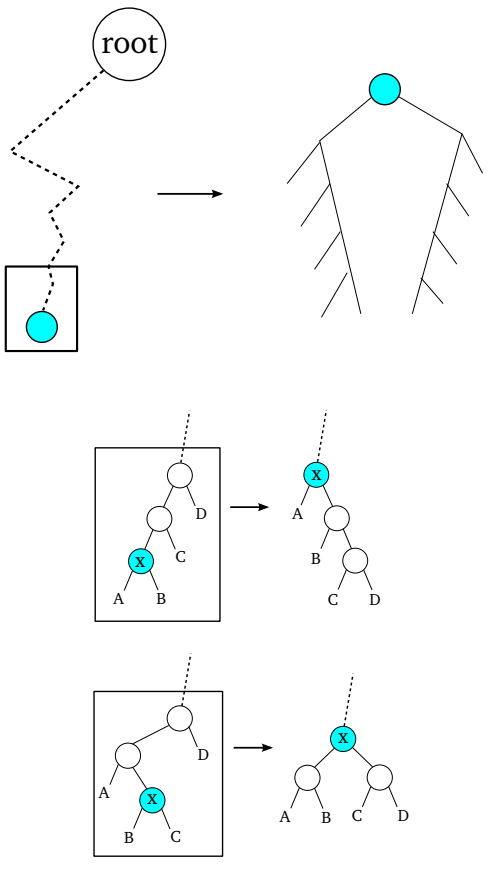

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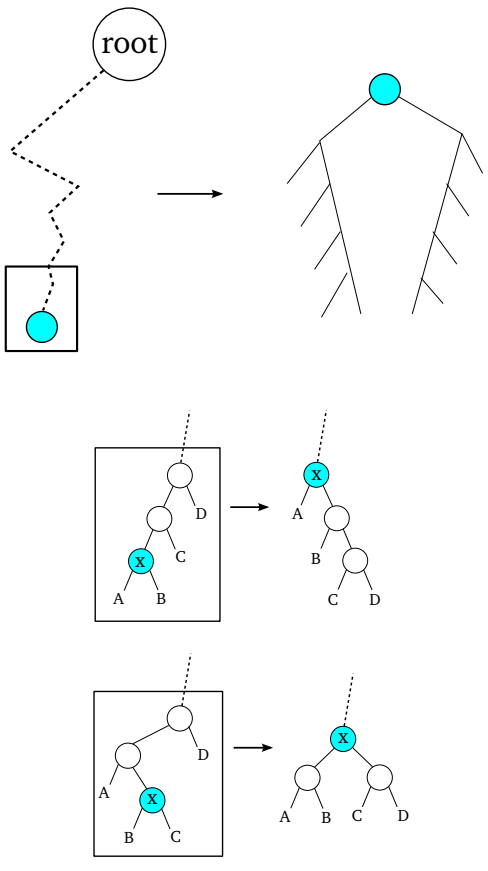

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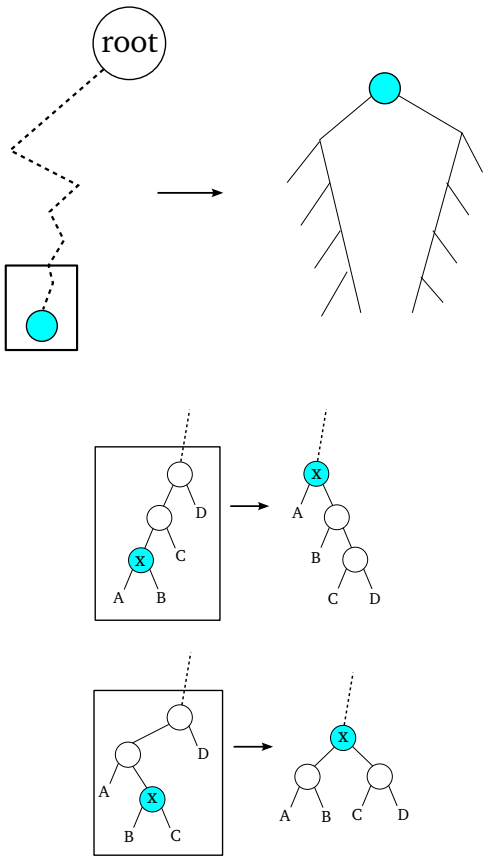

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- similarly for weighted case

Questions:

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[Subramanian, 1996; Georgakopoulos, McClurkin, 2004]:

A family of local transformations (templates):

- of constant size
- applied on the search path bottom-up
- fulfilling certain natural conditions

guarantee the access lemma (proof similar to splay proof).

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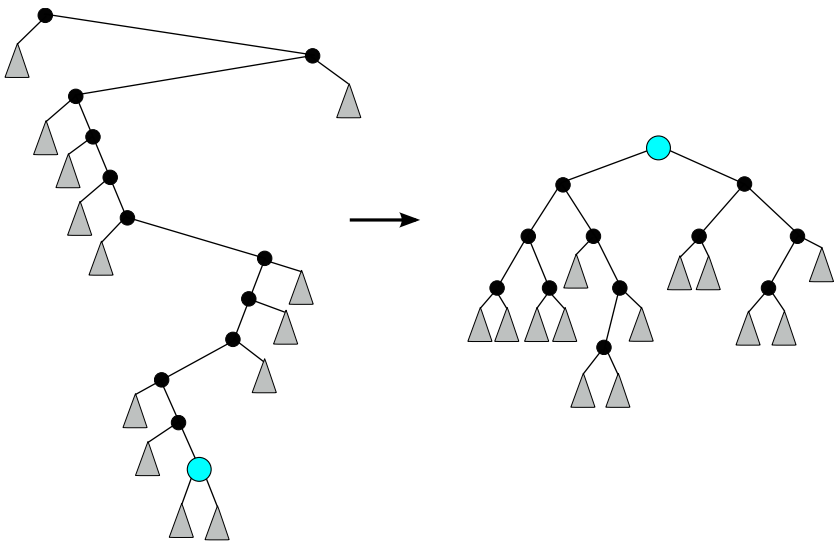

- (search path $\rightarrow$ after-tree) re-arrangement

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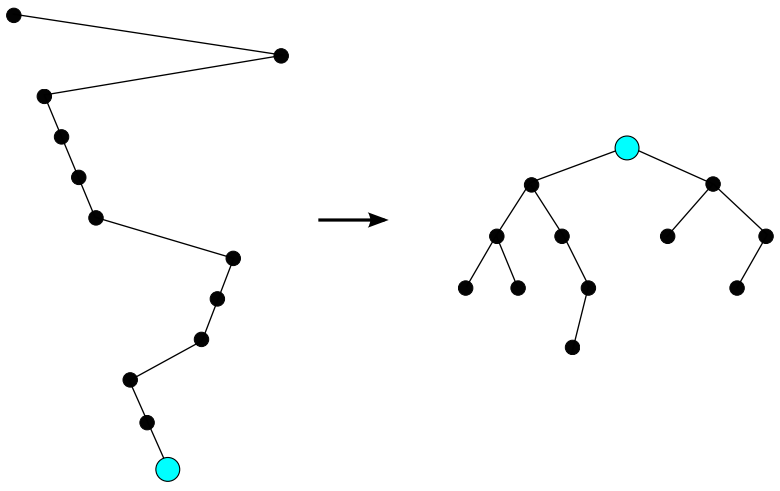

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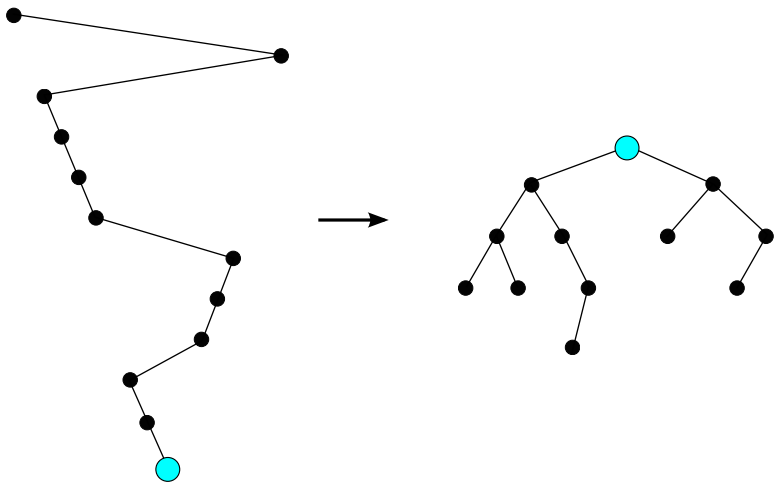

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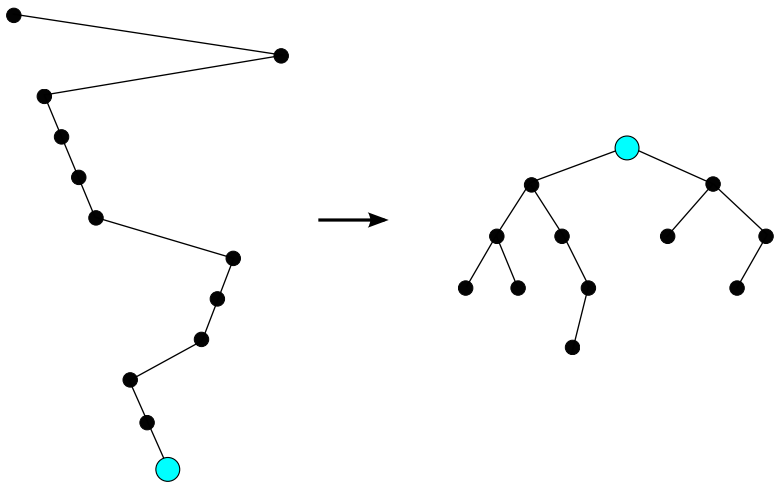
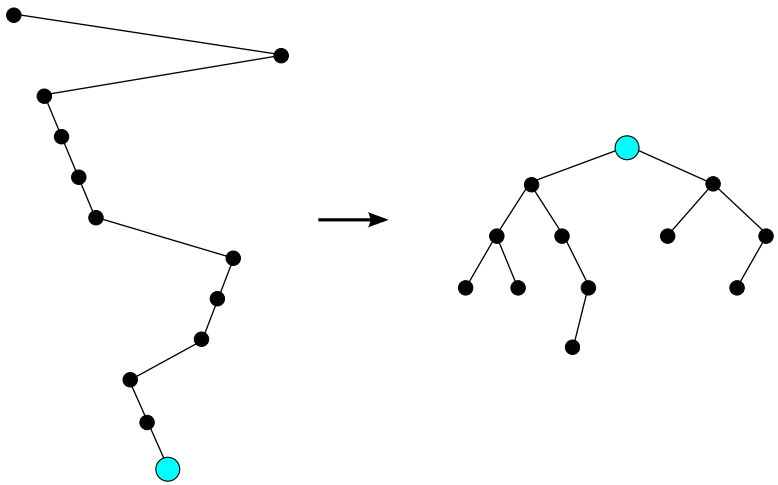


- (search path $\rightarrow$ after-tree) re-arrangement
- cost of re-arrangement is absorbed in access cost.
- **which re-arrangements lead to an efficient algorithm?**
- ideas from the literature: **path balance** (Sleator), **depth-halving** (Sleator, Tarjan, Subramanian, Georgakopoulos), ...

Properties of (search path $\rightarrow$ after-tree):



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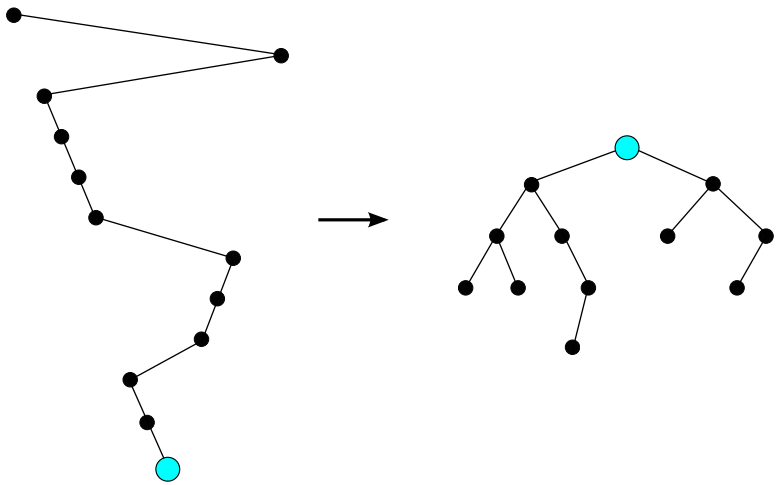

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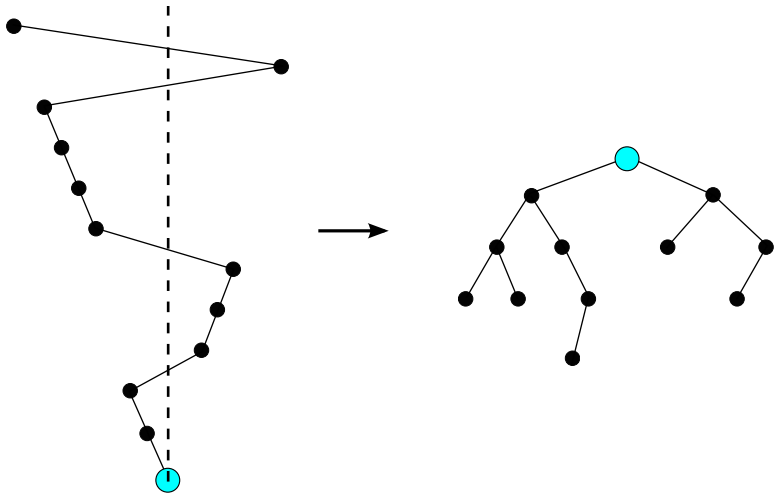

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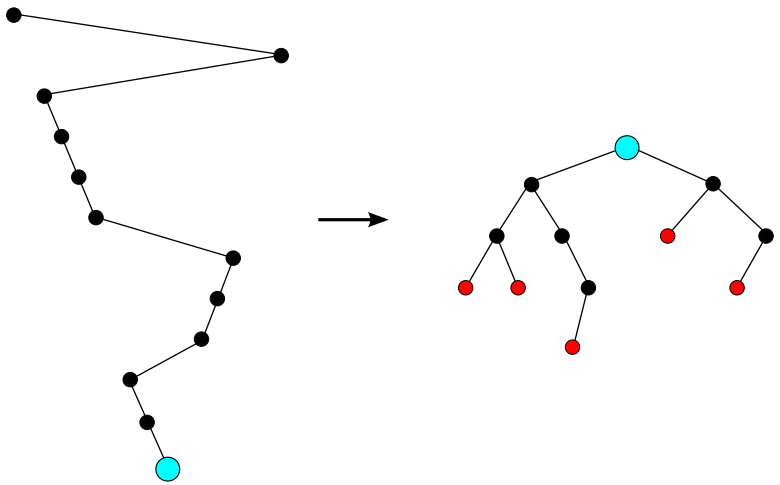

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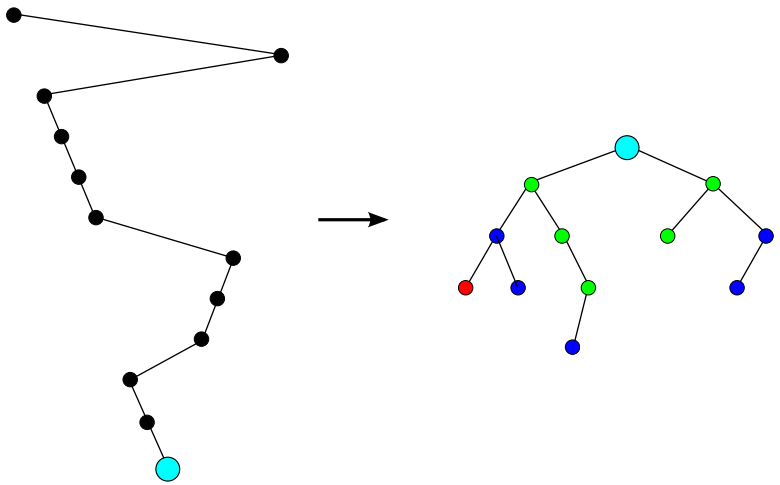

- length of the search path: $|P|$
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- max right- or left-depth: d

Main result (simplified):

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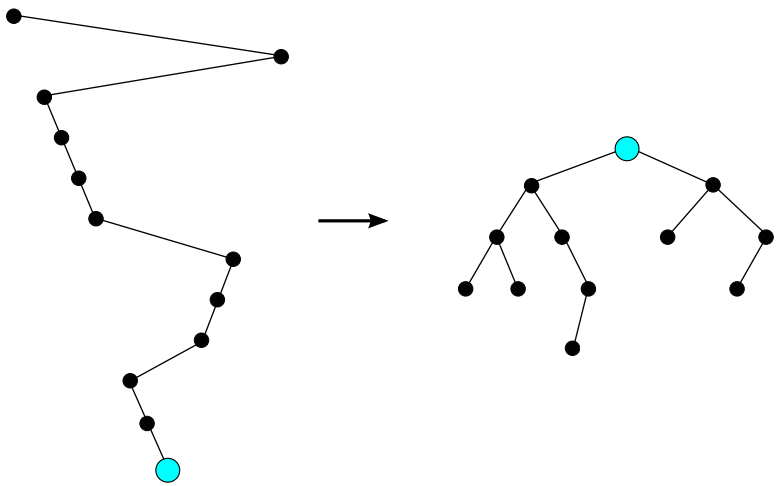

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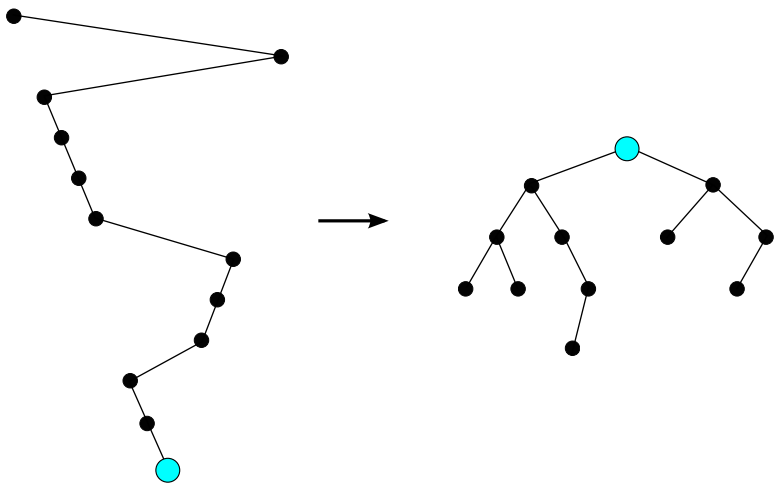

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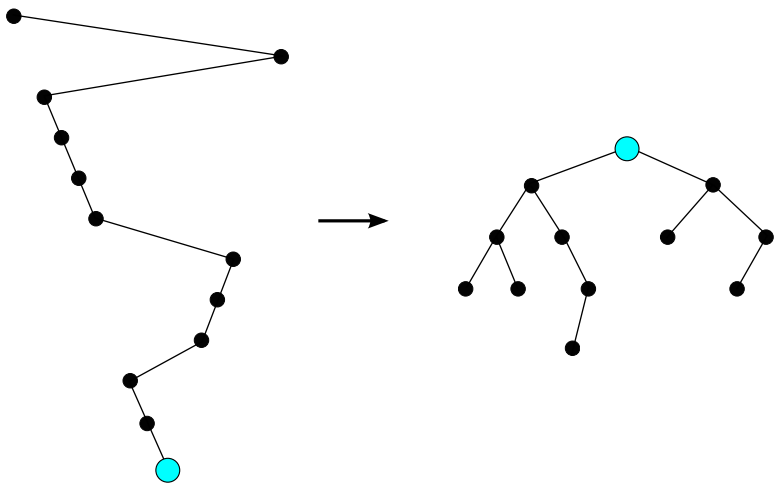

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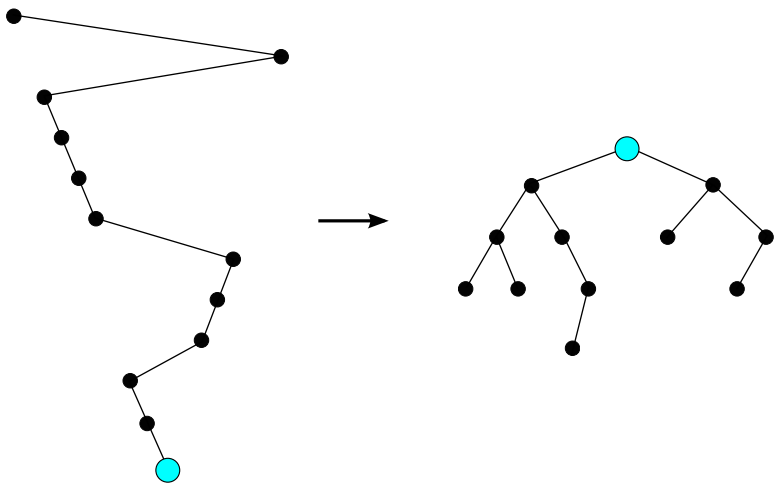

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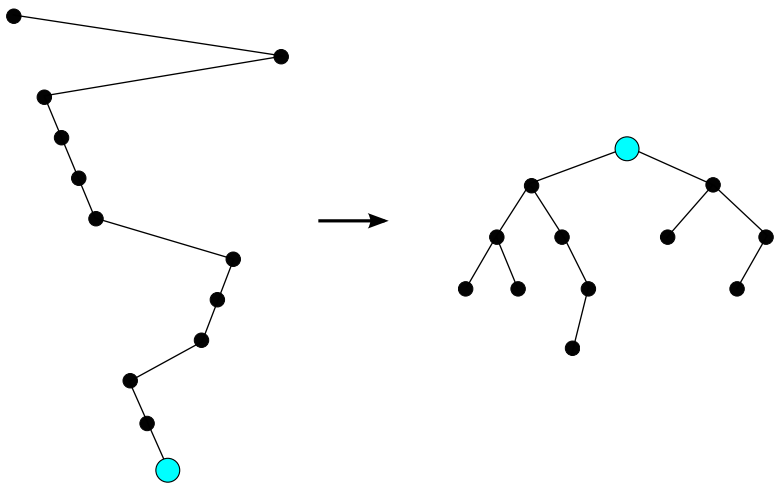

If

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$\Longrightarrow$ access lemma (weighted version).

Application 1: Splay Trees

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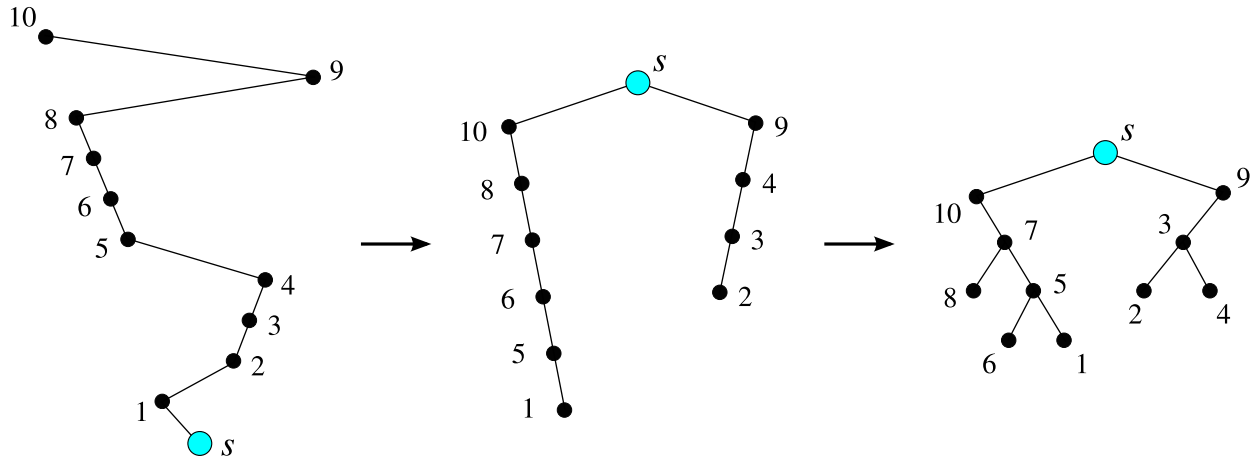

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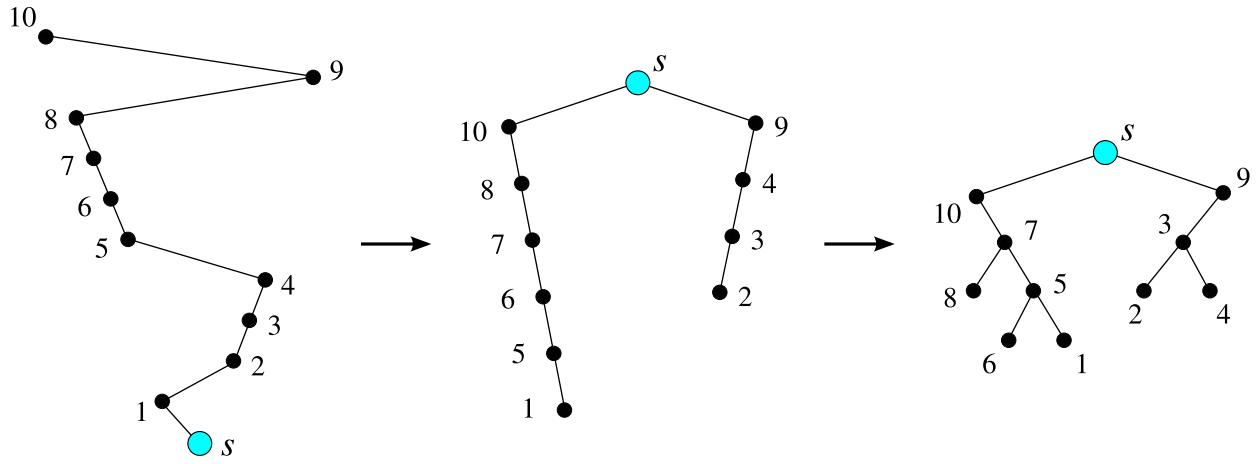

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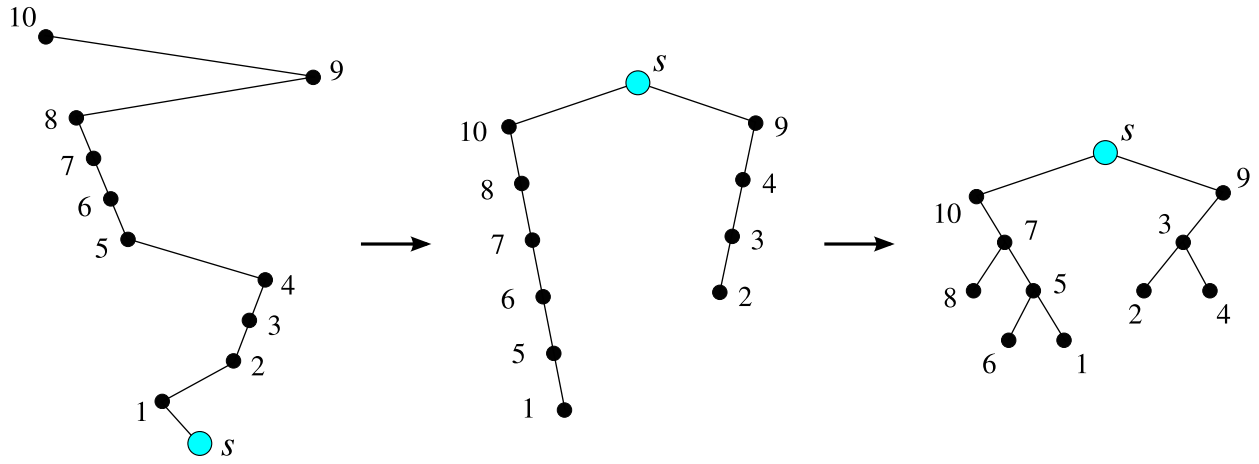

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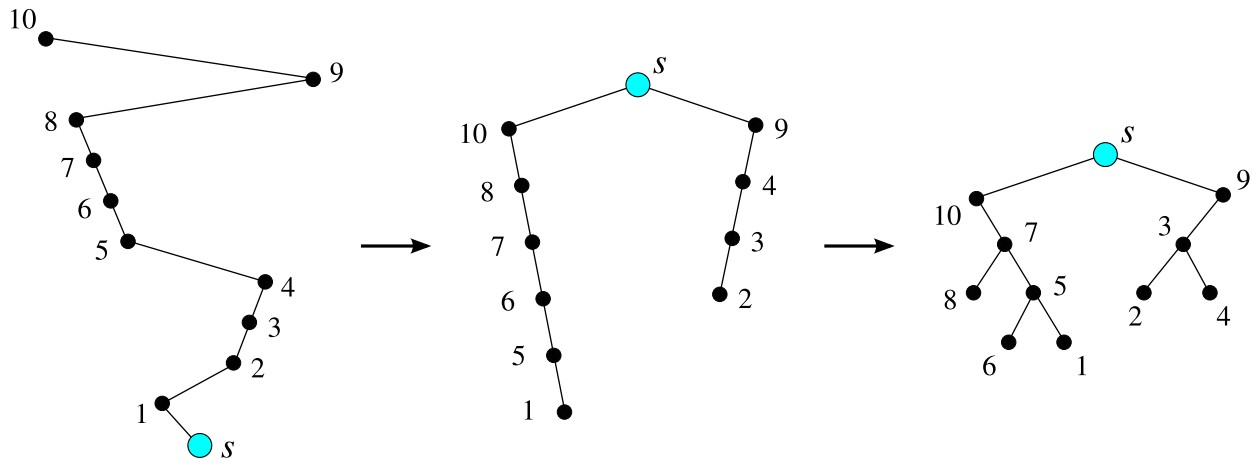

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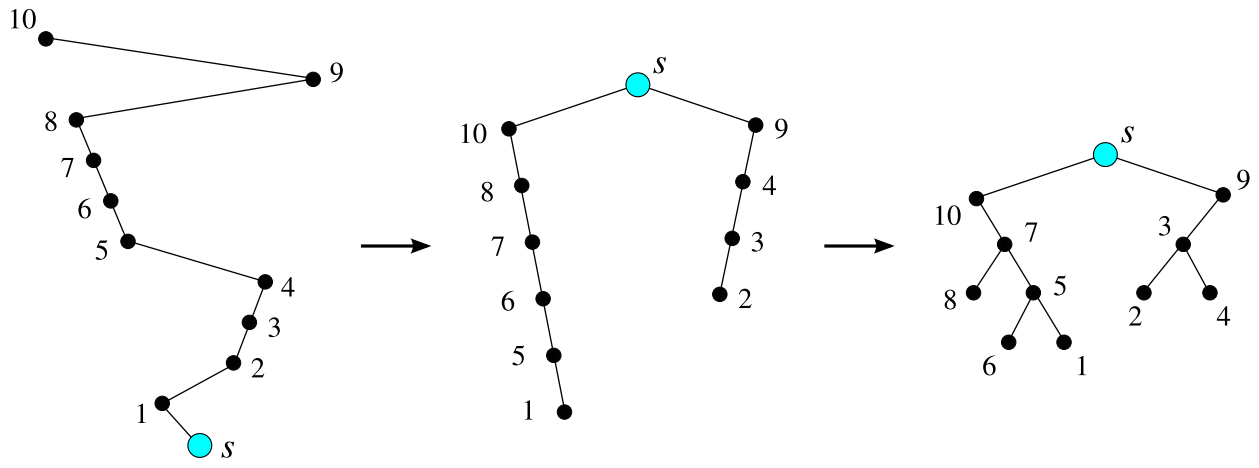

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Similarly for Subramanian's and Georgakopoulos and McClurkin's generalizations.

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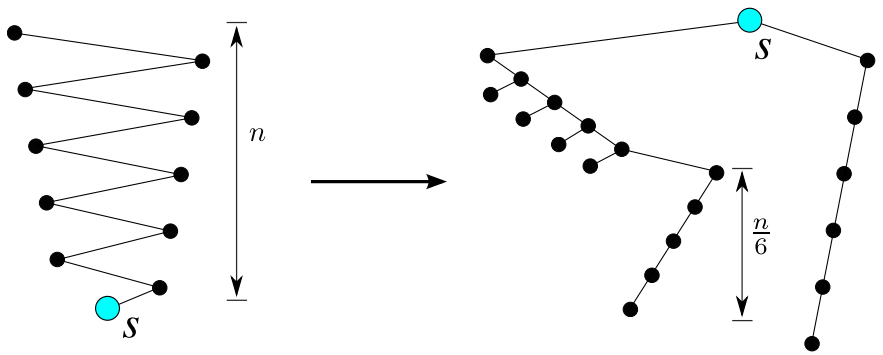

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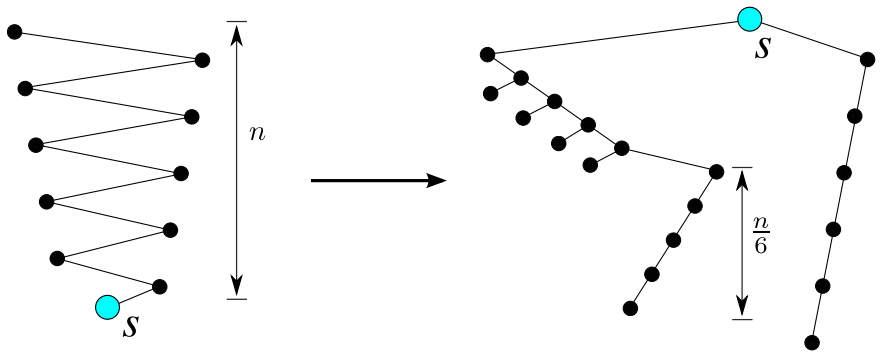

1. max left-depth is linear! $\times$

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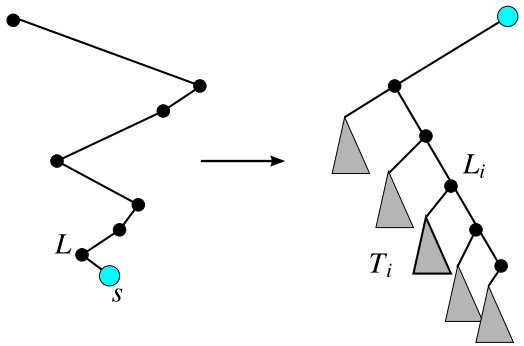

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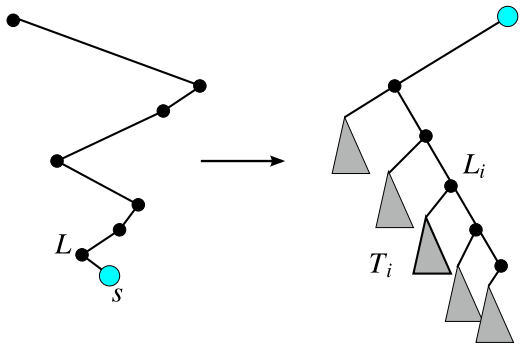

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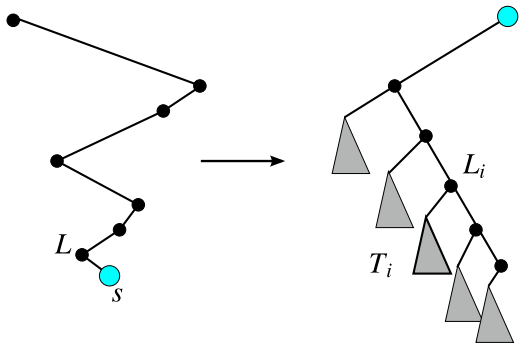

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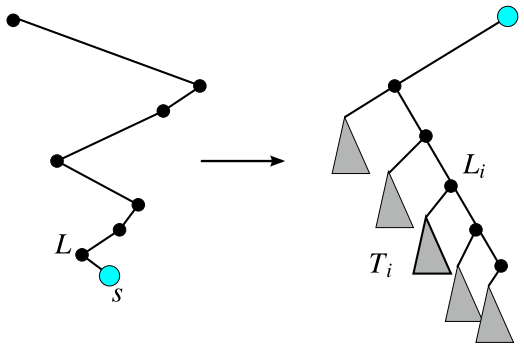

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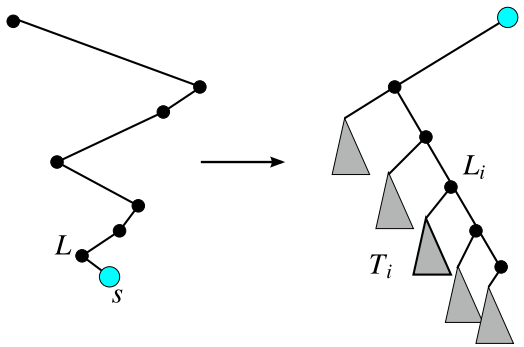

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Thus $(\epsilon d(s) - O(1))/O(1) = \Omega(d(s))$ trees T_i, each one having at least one leaf. ✓

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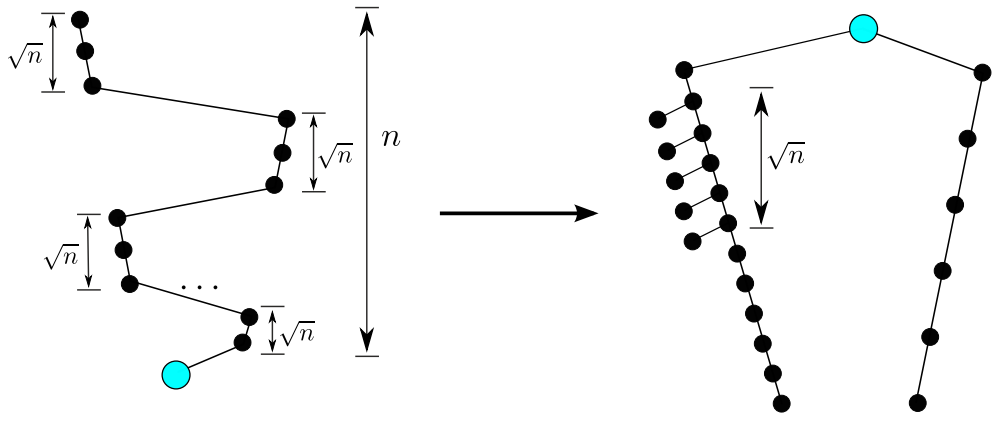

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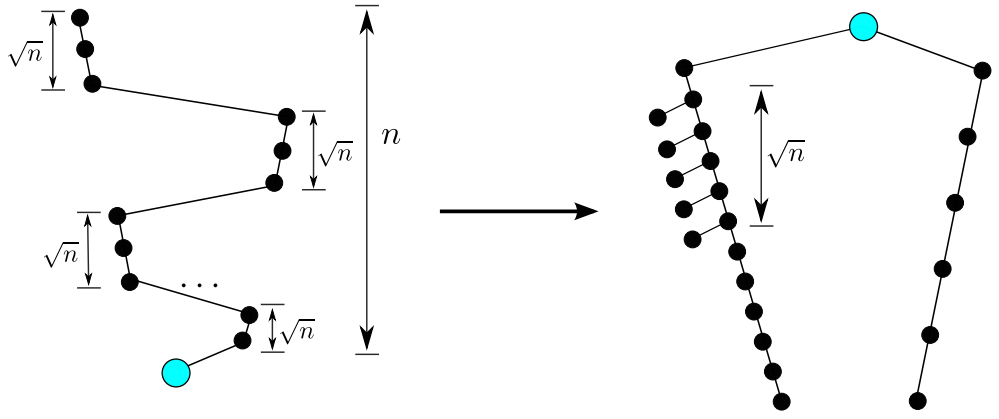

Open Question 1: Other natural heuristics for which the access lemma holds?

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Path Balance heuristic [Sleator]: transform search path into a balanced BST.

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A parameterized version of our theorem (simplified):

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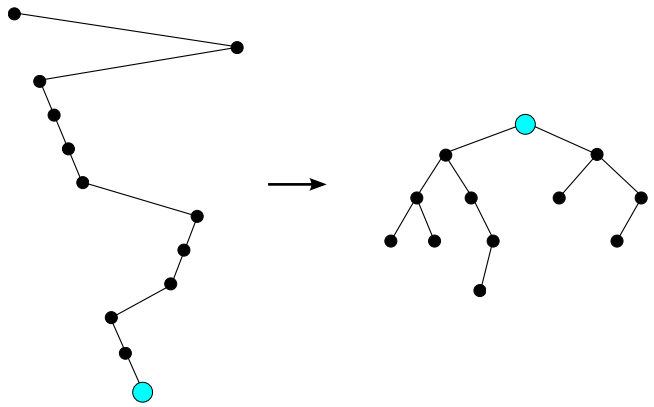
It follows that the total cost of m accesses is $O((m+n) \log n \log\log n)$.

Proof sketch of main theorem



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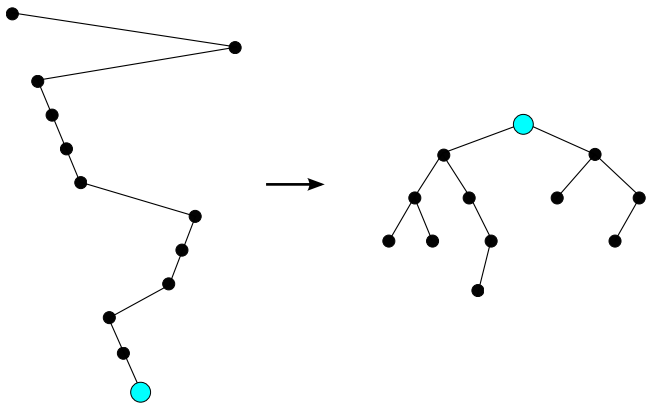

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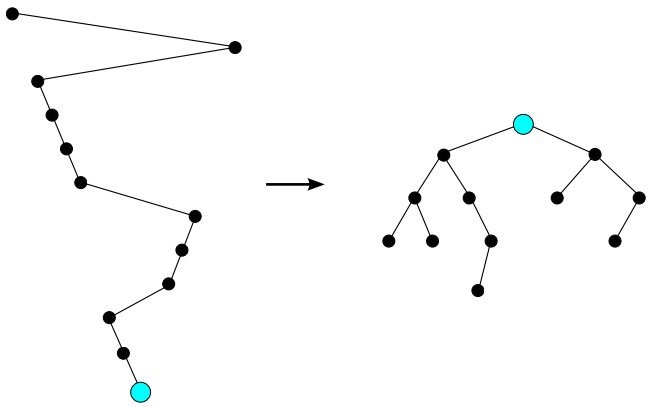

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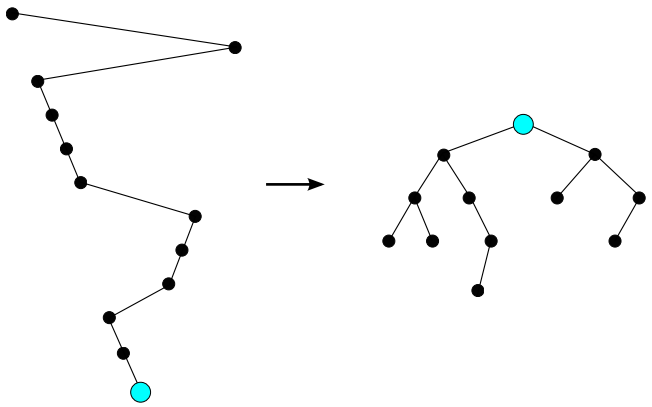

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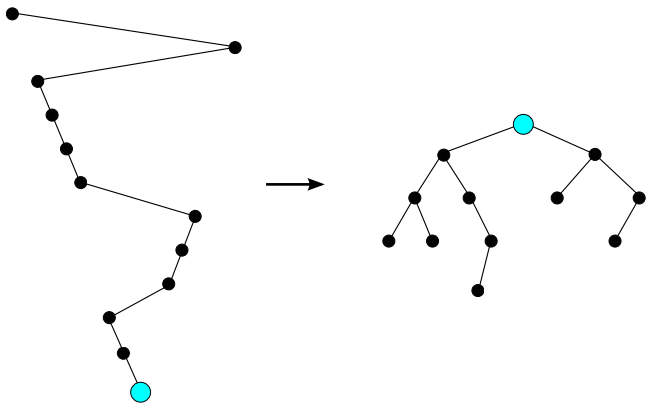

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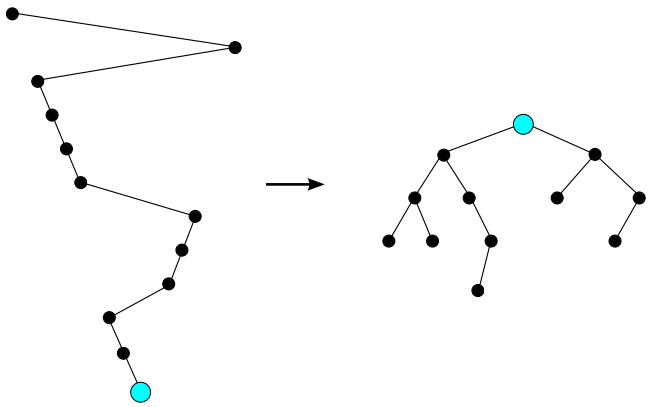

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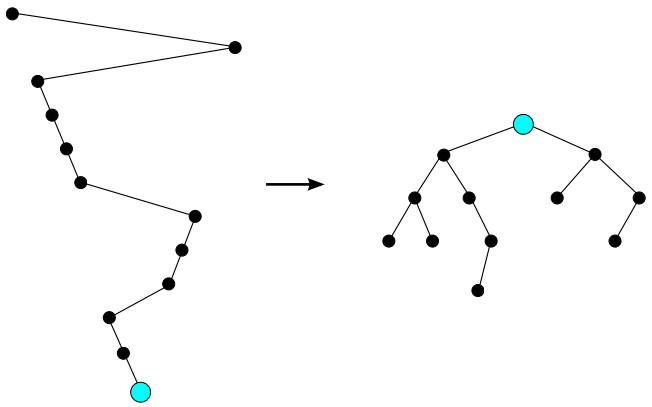

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If conditions hold: we can partition P into $O(1)$ neighborhood-disjoint and monotone sets $\implies$ access lemma.

Necessity of our conditions

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Proof idea: adversarial weight assignment.

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Necessity of (3)

If algorithm $\mathcal{A}$ satisfies (1) and (2), but only creates $n^{o(1)}$ leaves, then $\mathcal{A}$ needs $n \cdot \omega(1)$ time to access $1, 2, \ldots, n$.

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Open question 7: Which algorithms satisfy sequential access?

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Only the trivial $\log n$ approximation factor is known. Best factor for any algorithm: $\log \log n$ (Tango trees, Multi-splay trees, etc.)

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