Exact exponential algorithms for two post problems

[Lothar Koemps, Freie Universitaet Berlin, SWAT 2020]

Partially ordered set \( P = (X, \prec) \)

- ground set
- binary relation

\( \prec \)
  - transitive
  - irreflexive

Viewed as a (transitive) DAG

\[ \text{Def. Linear extension of } P = (X, \prec): \]

- total order on \( X \) that is consistent with \( \prec \)
- "contains \( \prec \)"

\[ \rightarrow \text{ a topological order of the DAG} \]

- Def. \( \# \text{LE} \) problem: Given \( P \), compute its number of linear extensions.

- Well-studied, fundamental problem:
  - optimal sorting
  - probabilistic ranking
  - \( \frac{1}{3} : \frac{2}{3} \) conjecture
  - \( \ldots \)

- \[ \text{[Llinel, 1986]} \]
- \[ \text{[Louiwi, 1986]} \]
- \[ \text{[Stanley, 1986]} \]
- \[ \text{[Dyier, Frick, Kawai, 1990]} \]

- \#LE is \#P-hard \[ \text{[Brightwell, Winkler, 1990]} \]

- \#LE can be solved in time \( O(n \cdot 2^n) \) \[ \text{[Similar DP as Bellman-Held-Karp for TSP]} \]

- Open Question: Solve \#LE in time \( O \left( (2-\varepsilon)^n \right) \) for some \( \varepsilon > 0 \)
**Dimension of a poset**

Def: \( \dim(P) \leq d \) if \( P \) can be embedded in \( \mathbb{R}^d \) s.t. "\( < \)" is "point domination":

\[
\begin{align*}
\forall a, b &
\quad a < b \iff a_i < b_i, \\
&
\quad \ldots \\
&
\quad a_d < b_d
\end{align*}
\]

Well-studied property [Trofim, 1992]

2-d posets already have rich structure:

\#\text{LE} is \#P-hard in 2-d posets [Ditmer, Pak, 2018]

**Question:** Solve \#\text{LE} in time \( O((2-\varepsilon)^n) \) in 2-d posets.

**Result:** \( O(1.32^n) \)

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**Standard DP:**

\[
\#\text{LE}(X) = \sum_{x \in \text{max}(X)} \#\text{LE}(X \setminus \{x\})
\]

\( \#\text{LE}(\emptyset) = 1 \)

Actual running time: \( \# \) reachable subsets = \( \# \) downsets of \( X \)
Suppose \( P \) has a large matching \( M \), \( |M| = \alpha \cdot n \)

matched pairs:

\[ x_i < y_i \]

\[ x_{|M|} < y_{|M|} \]

Obs. Downsets may contain both \( x_i, y_i \), neither \( x_i, y_i \), only \( x_i \), only \( y_i \)

\[
\# \text{downsets} \leq 3^{|M|} \cdot 2^{n-2|M|} = \left[ 2 \cdot \left( \frac{3}{4} \right)^{|M|} \right]^n
\]

(if \( M \) is large, we're O.K.)

Suppose \( P \) has no large matching (max matching \( M \), with \( |M| = \alpha \cdot n \))

→ Complement of \( M \) is an independent set = antichain

Antichain \( A \), \(|A| = (1-2\alpha)^n\)

- \( 2n - \alpha n \) points in \( M \) split \( A \) into \( \leq 4\alpha n \) groups

- Points in a group are indistinguishable
Group sizes $n_1 + n_2 + \ldots + n_e = (1-2\xi)n$

Obs. In a reachable state we need to know # points from each group, but not the exact subset.

Replace groups by chains of same size (connections to $M$ preserved)

Solve #LE in modified Poset (correct by factor $n_1! \cdot n_2! \cdot \ldots \cdot n_e!$)

$$
\#\text{downsets} = \prod_{i=1}^{e} (1+n_i) \cdot 3^{\alpha n} \leq \left[ \frac{(1-2\xi)n + \ell}{\ell} \right]^{\ell} \cdot 3^{\alpha n}
$$

$$
\sum_{i=1}^{e} (1+n_i) = (1-2\xi)n + \ell
$$

(product is max. if all terms equal)

$$
\leq \left[ \frac{(1-2\xi)n + 4\alpha n}{4\alpha n} \right]^{4\alpha n} \cdot 3^{\alpha n}
$$

$$
= \left[ \frac{1+2\xi}{4\alpha} \cdot 3^{\alpha} \right]^{\alpha n}
$$

Summary:

- Using matching only: $2 \cdot \left( \frac{3}{4} \right)^{\frac{\alpha n}{\ell}}$
- $\alpha = \frac{1}{6}$
- Using antichain: $\frac{1+2\xi}{4\alpha} \cdot 3^{\alpha}$

Graph:

- antichain
- matching only

1.91^n
Improvements:

I. Unmatched edges create only 2xh groups (if M chosen carefully)

II. Packing larger subgraphs

More tricks:
- Pack larger subgraphs (4,5,...)
- Combine matched edges into quartets
- Split M into groups
- ...

Open Question: Solve #LE in time $O((2-\varepsilon)^n)$ when $\dim(p) \geq 3$. 
Jump # problem

\[ |k_{t^*} - k_{t^*}| \rightarrow \text{slight improvement } O(1.82^n) \]